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DETERMINISTIC AND STOCHASTIC DYNAMICS WITH HYPERBOLIC HJB-TYPE EQUATIONS

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Abstract. The study of deterministic and stochastic dynamic problems in control theory can be reduced to partial differential equations by using Bellman's approach. The resulting PDE-based mathematical models are difficult to solve numerically, and most existing approaches cannot provide accuracy expected in practical applications. It is proposed to approximate the original controlled dynamics with a sequence of PDEs. By applying the Steklov-Poincare operator technique the general form of equations in such a sequence has been established. The derived model can be used in cases that are not covered by standard diffusion processes.
Keywords. Hyperbolic PDEs, Steklov-Poincare averaging, jumping Markov processes.
AMS (MOS) subject classification: 35L10, 49L20, 65C50.

1 Introduction

Dynamics, both deterministic and stochastic, should be controlled in many applications ranging from industrial processes to problems in life sciences and financial market (e.g., [16, 8, 27]). The problem of control can be associated with a Hamilton-Jacobi-Bellman (HJB) equation, and with fundamental advances in viscosity solution theory and nonsmooth analysis there is a substantial body of literature in this direction [32, 29, 26, 18, 17, 2, 3, 6, 11, 12, 13, 5, 1]. One of the most important practical reasons for this interest to the HJB equation lies with the fact that when the value function is smooth, apart from giving the classical solution of that equation, the derivatives of this function allow us to construct optimal strategies in feedback form. It is well known, however, that the value function should be considered in some generalised sense (e.g., as contingent solution, semicontinuous viscosity solution, minimax solution, etc [5, 27]). In addition, the task of solving numerically the resulting equation remains extremely difficult and most available methodologies are yet to be improved in terms of accuracy in order to be applicable in realistic practical situations [17]. The problem becomes even much more difficult when control is a function of both temporal and spatial variables.

We recall that the coupling between position $x \equiv q$ and momentum p is an essential ingredient of both deterministic and stochastic dynamics. In

HJB-based models this coupling is realised via the system Hamiltonian [19]. A similar situation is observed in stochastic (quantum) mechanics where one has to deal with two stochastic differential equations, one describing the Nelson process $x \equiv q$ and the second describing the momentum. Like in the classical case, this system can also be re-written in the form of the canonical system, but with some perturbations [24]. If the momentum field can be represented as $p(\xi, t) = \nabla S_q(\xi, t)$, it is possible to derive a HJB-type equation with respect to function S_q (action, cost, or value function) which is dependent on the dynamics of the process and control. The fact which is often overlooked is that the form of the HJB equation could be dependent *strongly* on the dynamic rules for the probability density of the process $(q(t), p(t))$. Such dynamic rules are usually described by the Fokker-Planck-Kolmogorov equation which is *local* in time and space [25]. The situation is similar for control models where the HJB equation, written with respect to the cost (or value) function, should not be considered in isolation, but rather to be coupled to the dynamic rules for the probability density, i.e. to *another evolution equation.* Traditionally, these two equations have been modelled by the parabolic-like dynamics [9], and despite important contributions to the development of hyperbolic stochastic differential equations (e.g., [10]) the connection between hyperbolic dynamics and control problems has not yet been explored with vigour it deserves. However, if we need to maintain long-range order in systems governed by short-range interactions, the associated models based on the HJB equation and the probability density evolution becomes strongly coupled, the fact that causes substantial difficulties in the numerical solution of such models. This situation arises in many applications of life sciences as well as in reaction-diffusion systems considered in a bounded domain and driven by space-time fluctuations. In this situation we have to deal with *interactive dynamics between macroscopic and microscopic scales.* A general framework for dealing with such problems encountered in stochastic mechanics was laid in [24] where the total velocity of the system was interpreted as a complex-valued stochastic process including unperturbed (current) velocity and perturbed (osmotic) velocity. In the context of control problems a general framework for dealing with strongly coupled systems was developed in [19, 20] where the total velocity was also split into two parts, but both parts were subject to perturbations. For the regularisation of mathematical models of control based on strongly coupled dynamics of control and probability density the author in [19] proposed to use the conjugate pair of probabilistic weights. The approach allows us to study complex controlled dynamic systems, the evolution of which could exhibit hyperbolic behaviour and could be non-local. The main purpose of this paper is to develop a procedure that would allow to split HJB-type equations encountered in these applications into a sequence of equations and to derive a general form of equations in this sequence. Then, the original controlled dynamics can be *sequentially* approximated, and at each step of this approximation the solution can be interpreted by using generalised solution theory.

Finally, we note that such an approximation can be constructed by using the Markov chain approximation method [16] with consistency conditions discussed in [20, 22]. In this case the original controlled dynamics can be interpreted as a stochastic process in the Davis-Jacod-Skorokhod sense (e.g., [14]).

2 Drift-Diffusion Models of Controlled Dynamic Systems

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, and let $W_p^1(\Omega)$ be the Sobolev space consisting of those absolutely continuous functions $u : \Omega \to \mathbb{R}$ that $\partial u/\partial x_i \in L^p(\Omega)$, $i = 1, ..., n$, $p \in [1, \infty]$. Then, recall that $W_1^\infty(\Omega) \equiv \mathrm{Lip}(\bar{\Omega})$, $W_1^1(\Omega) \equiv AC(\bar{\Omega})$, and $W_q^1(\Omega) \subset W_p^1(\Omega)$ for all $p \leq q$. In a similar manner (e.g., [28]), we define class $W_p^{m,n}(Q_T) \equiv W_p^m(I; W_p^n(\Omega))$ in the time-space region $Q_T = I \times \Omega$ with $I = (t_1, t_2)$ for $t_1 < t_2$ (e.g., $t_1 = t_0$ and $t_2 = T$), and with m and n being responsible for function regularities with respect to time and space, respectively.

In a quite general setting the problems of interest in this paper can be formulated as the minimisation problem for a certain functional over $x \in W_1^1([t_0, T]; \mathbb{R}^n)$ satisfying

$$J(x(\cdot)) \to \min, \quad x'(t) \in F(t, x(t)) \quad \text{a.e.} \tag{1}$$

$$x(t) \in G \quad \forall t \in [t_0, T], \quad (x(t_0), x(T)) \in C_0 \times C_1, \tag{2}$$

$G, C_0, C_1 \subset \mathbb{R}^n$ are nonempty and closed, F is locally Lipschitz, and J is a given functional (e.g., $J(x(\cdot)) = k_0(x(t_0), x(T))$ with $k_0 : \mathbb{R}^{2n} \to \mathbb{R}$). As usual, F in (1) is a multifunction, e.g. in the case of state constraints under quite general assumptions (e.g., [27]) one can set

$$F(t, x) = \{\tilde{f}(t, x, u) : u \in \mathcal{U}(t, x)\}, \tag{3}$$

where $\mathcal{U}$ is a state-dependent constraint set, a set intersection between a set of classical, possibly control dependent, state constraints and the classical control set, e.g., [12]. So that the problem (1)–(3) is equivalent to a conventional formulation with the dynamics defined as

$$\dot{x}(t) = \tilde{f}(x, t, u), \quad u \in \mathcal{U} \quad \text{a.e.} \tag{4}$$

For such problems (although originally without state constraints) the Pontryagin maximum was the historically first tool to apply. Note further that since (1)–(3) (and (4)) is understood in almost everywhere sense, in a number of important cases function $\tilde{f}$ can, under appropriate conditions, accommodate both drift (f) and diffusion (σ), pertinent to the stochastic controlled dynamics considerations

$$x(t) = x(t_0) + \int_{t_0}^{t} f(\tau, x(\tau), u(\tau))d\tau + \int_{t_0}^{t} \sigma(s, x(s), u(s))d\omega(s), \tag{5}$$

where $\omega(\cdot)$ is, e.g., a Wiener process defined on a corresponding (possibly filtered) probability space. In such cases the performance measure functional should be understood in the probabilistic sense of mathematical expectation

$$J(u) \equiv E^u_{x,t}[\tilde{J}(u)] \to \min, \tag{6}$$

where, for example, for the Boltza problem $\tilde{J}(u) = \int_{t_0}^T f_0(\tau, x(\tau), u(\tau))d\tau + \alpha g(T, x(T))$ with given f_0, g, and α.

It is important to note that the considerations (1) – (3) and (5) – (6) are not necessarily should be limited to the diffusion type stochastic differential equations only. If both controlled and uncontrolled diffusions are included, we can combine them in function $\tilde{\sigma}^2$, and then choose the model for drift in such a way that $\int_{t_0}^T \tilde{\sigma}^2(t, x, u)dt = 0$ subject to control $u(t, x)$. Indeed, $\tilde{\sigma}^2(t, x, u(t, x)) = 0$ a.e. in $[t_0, T)$ if and only if $\tilde{\sigma}^2(t, x, u(t, x)) \in L^1$ and $\int_{t_0}^T \tilde{\sigma}^2(t, x, u)dt = 0$. More precisely, we can introduce a set $M(\mathcal{U})$ of probability measures on $\mathcal{U}$ in a way similar as described in [12]. Then, if $M(\mathcal{U}) \subseteq [C(U)]^*$, $\hat{U}(t_0, T) = \{\mu \in L^\infty([t_0, T]; M(\mathcal{U}))$ for a.e. $t \in [t_0, T], \mathrm{supp}(\mu_t) \subset G(t, x(t))\}$ is the space of admissible relaxed controls (G is a set-valued map $[t_0, T] \times \mathbb{R}^n \to \mathcal{U}$, defined by our state-dependent constraints, so that $G(t, x) = \{u \in \mathcal{U}\}$), and $x(\cdot)$ is the unique solution to the equation

$$x(t) = x(t_0) + \int_{t_0}^t \int_{\mathcal{U}} f(s, x(s), u)\mu(s, du)ds, \tag{7}$$

we have that $x(\cdot)$ is absolutely continuous, and is known as the generalised curve of (7), e.g., [13]. Hence, the starting point of our consideration is a sequence of problems

$$x(t) = x(t_0) + \int_{t_0}^t f(t, x(t), u(t, x))dt + \int_{t_0}^t \tilde{\sigma}^2(t, x(t), u(t, x))dt \tag{8}$$

which, under assumptions made, is reducible to a sequence of differential inclusions (1) subject to an appropriate partition of interval $[t_0, T]$ (see Section 1). The models of the form (8) describe general dynamic systems and have two main features: (a) state-dependent constraints, incorporated into the model via the L^1-function $\tilde{\sigma}^2$, and (b) space-time dependent controls which couple both parts of the model (drift and diffusion), describing dynamic properties of the system. As a result, constraints of the problem and its controls are intrinsically coupled, and a key to the understanding of the dynamics described by model (8) is kept by the time interval partition dependent on the goal functional $J(u)$. It is this partition that brings elements of randomness into the model. Viewing the dynamic system description as a map $\Sigma \to \Sigma'$, and constructing a model for system dynamics based on differential/integral models (as well as models based on other tools) should reflect a possibility of an error, δ, in the definition of the initial conditions which might lead to perturbations of the functional (e.g., the Hamiltonian)

on which the construction of the model is based. If the dynamics of the system is to be controlled, *such perturbations should be reflected in the system functional at the stage of the model construction, rather than at the stage of the problem solution.* We denote such a perturbed functional by H_ϵ^δ, where ϵ reflects changes in perturbations of the system functional with changes in δ. Since the level (or dynamics) of perturbations of the dynamic system is not known a priori, we expect that the dependency $\epsilon(\delta)$ might be correlated with the dependency between the goal functional $\tilde{J}(u)$ and control $u(t,x)$. Under these circumstances, it is more appropriate to deal with local vector fields, e.g. $(v(t,x), \epsilon(\delta))$, rather than with single-valued functions, e.g. $v(t,x)$, respectively. The classical consideration would follow from our consideration in the limit as $\epsilon(\delta) \to 0^+$. The later condition, however, may not always be satisfied [19].

Problems like (1)–(3) and (5)–(6) can exhibit quite complicated behaviour, including situations where standard assumptions typically used for their analysis may fail. Indeed, consider a simple problem of minimisation over class $W^{1,1}([a,b];\mathbb{R}^n)$, i.e. over the Banach space of absolutely continuous $\mathbb{R}^n$-valued functions on the interval $[a,b]$ with norm $\|x\|_{W^{1,1}} \equiv |x(a)| + \int_a^b |\frac{dx(t)}{dt}|dt$, and take $n=1$, $\Omega \equiv (a,b)$ for $a<b$,

$$J(u) \equiv \int_a^b f(x,u,u')dx : W_1^1(a,b) \to \mathbb{R} \cup \{+\infty\}, \tag{9}$$

say, assume that $f \in C^\infty([a,b] \times \mathbb{R} \times \mathbb{R}), f(x,y,z) \geq 0 \quad \forall x \in [a,b], y,z \in \mathbb{R}$ for simplicity. Then the Lavrentiev phenomenon can be pronounced in the existence of singular minimisers where one has to deal with AC functions which might not be Lipschitz continuous (e.g., [4, 13, 7]). Although conditions exist that allow to verify whether this phenomenon is generic or not, for the class of control problems with state constraints, formulated at the beginning of this section, the appropriate classes for the analysis of these problems become those Sobolev classes $W_p^{m,n}$ where $p=1$ [21].

Note further that for problems like (5)–(6) Bellman's dynamic programming approach is quite natural to apply. The connection between the two major approaches to control problems, the Pontryagin maximum principle and the Bellman approach, has been a subject of intense investigations (e.g., [2, 30, 31, 29, 27] and references therein). This connection has been studied extensively for non-smooth problems which can be cast in form (1) – (3). If we define the value function V as the greatest-lower-bound performance measure to the problem, $V(t,x) = \inf_{u \in \mathcal{U}} J(t,x;u)$, then in the classical case the connection between the value function V of the HJB partial differential equation [19] $\partial V/\partial t + H_\epsilon^\delta(t,x,\partial V/\partial x) = 0$, where $H_\epsilon^\delta(t,x,\partial V/\partial x) = \sup_{u \in \mathcal{U}}(-f_0 - f\partial V/\partial x)$, and the adjoint function, $\psi^p(t)$, of the Pontryagin principle is given by a simple equality relationship $\partial V/\partial x(x^*(t),t) = \psi^p(t)$ with the optimal solution pair (x^*,u^*), provided $V \in C^{1,2}(Q_T)$ (e.g., [15]).

However, in the general (and more realistic) case, the connection can be established in terms of inclusions only (e.g., [30, 31, 29] and references therein). The major difficulty in the analysis lies in the definition of the system functional (e.g., the Hamiltonian for dynamical systems) which in the general case *should be defined in terms of local vector fields, accounting for the dynamics of* $\epsilon(\delta)$, as we explained above. In the classical setting of dynamical systems, one would need some *additional conditions* on the Hamiltonian to proceed with the analysis of the associated control problem. Indeed, consider the classical Dubovitskii-Milutin example with $J(u(\cdot)) = k_0(x(T))$ and constraints $x(t_0) = x_1, \quad g(x(t) \leq 0 \quad \forall t \in [t_0, T]$ (see [1], and also [13]). In this, relatively simple, case one can define the Hamiltonian of the system as $H_\epsilon^\delta(x, t, \psi^p) = \sup_{u \in \mathcal{U}} \mathcal{H}(x, t, u, \psi^p)$ where $\mathcal{H}(x, t, u, \psi^p)$ is the Hamilton-Pontryagin function. If, as before, the pair (x^*, u^*) solves the problem, then it is important to realise that among other conditions, the condition of absolute continuity of function $h(t) = H_\epsilon^\delta\left(x^*(t), t, \psi^p(t) + \int_{t_0}^t d\eta\right)$ where, e.g., ψ^p is a left-continuous BV function and η is a nonnegative bounded Borel measure on $[t_0, T]$ with $||\eta|| = \sup_{||x||_{C(I)}=1} \int_{t_0}^T x(s) d\eta$, and, what is even more important, the jump condition for the Hamiltonian

$$H_\epsilon^\delta\left(x^*(t), t, \psi(t) + \int_{t_0}^t d\eta\right) = H_\epsilon^\delta\left(x^*(t), t, \psi(t) + \int_{t_0}^t d\eta - \eta(t)\right) \qquad (10)$$

$\forall t \in [t_0, T]$ are essential. Certainly, specific types and forms of jump conditions that should be imposed on the system functional (e.g., the Hamiltonian) in a more general case are subject to the choice of the measure on $[t_0, T]$, but an important point to emphasise that the conditions of these types are notably absent in the analysis of the HJB models. In this situation it is natural to seek developing an approach that allows us to *constructively approximate the system Hamiltonian*, and to split the original problem into a sequence of smaller problems.

3 Pontryagin-Hamilton's Representation of Coupled Dynamic Controlled Systems and Its Sequential Approximation

The importance of normalisation in the choice of the Hamilton-Pontryagin function has been discussed in detail in [19]. Without going into details, here we only note that instead of considering the control problem in terms of trajectory-control pair (x, u) it is often useful to consider a triple $< \psi_1^u | H_\epsilon^\delta | \psi_2^u >$ with the perturbed (due to state/control constraints) Hamiltonian of the system H_ϵ^δ, and with two functions $\psi_1^u(t, x)$ and $\psi_2^u(t, x)$ participating in the definition of local vector fields and are defined as spatial and temporal averagings of the value function, respectively [19, 21]. This

consideration leads us to a set of modified canonic equations $\partial\psi_1^u/\partial x = H_\epsilon^\delta/\partial t$, $\partial\psi_2^u/\partial t = -H_\epsilon^\delta/\partial x$, and to an alternative characterisation of the control problem by the triple $< \psi_1^u|Q^u(t,x)|\psi_2^u >$ with $Q^u(t,x)$ being a performance measure (defined below) which is coupled to the Pontryagin-Hamilton function, and, therefore, to a specific form of normalisation (see [19] for further details). All functions in the discussion that follows are assumed to be at least from $W_1^{1,1}$. Now, our main goal is to derive an evolution equation for Q^u.

The performance measure is introduced for any arbitrary interval $[t,T]$ with $t_0 \le t \le T$ as follows (we consider for definiteness the Boltza problem with $\alpha = 1$)

$$Q(t,x;u(x,\tau),t\le\tau\le T) = \int_t^T f_0(\tilde\tau, x(\tilde\tau), u(\tau,x))d\tilde\tau + g(x(T),T), \quad (11)$$

i.e. it is introduced by a memory-function which is *dependent on the control history*. As in the standard Bellman approach, the task is to find an optimal performance measure (the minimum cost function) defined by

$$Q^u(x,t) = \min_{u(x,\tau)\in U,\, t\le\tau\le T} Q(\tau,x,u(x,\tau)). \quad (12)$$

Choosing a time-step Δt, from (11) and (12) we obtain

$$Q^u(x,t) = \min_{u(x,\tau)\in U,\, t\le\tau\le T} \left\{ \int_t^{t+\Delta t} f_0 d\tau + \left(\int_{t+\Delta t}^T f_0 d\tau + g(x(T),T) \right) \right\}. \quad (13)$$

Then, the idea of *sequential approximations* requires [15]

$$Q^u(t,x) = \min_{u\in U,\, t\le\tau\le t+\Delta t} \left\{ \int_t^{t+\Delta t} f_0 d\tau + Q^u(t+\Delta t, x(t+\Delta t)) \right\}, \quad (14)$$

where $Q^u(t+\Delta t, x(t+\Delta t))$ represents the minimum cost of the controlled process in the time interval $[t+\Delta t, T]$. The accuracy of such approximations depend on regularity requirements imposed on function Q^u. In the classical case the essential assumption used to obtain an approximation based on (14) is that the limit $\delta = \lim_{\Delta t\to 0^+} R/\Delta t$, where $R = \frac{1}{2}\left\{\frac{\partial^2 Q}{\partial x^2}(\Delta x)^2 + 2\frac{\partial^2 Q}{\partial x\partial t}\Delta x\Delta t + \frac{\partial^2 Q}{\partial t^2}(\Delta t)^2\right\}$, is vanishing (see details in [19]). Recall that for both deterministic and stochastic cases a formal derivation of the HJB equation can be obtained with Taylor's series expansion. This leads to a priori excessive assumptions on smoothness of the value function. These assumptions were removed by the development of viscosity solution theory and nonsmooth analysis techniques [6, 18, 26, 5, 27]. However, it is instructive, and important from both theoretical and practical points of views, to give a formal procedure of obtaining a sequential approximation, resulting from (14) under minimal smoothness assumptions. In the next section we analyse the derivation of this procedure by applying the Steklov-Poincare technique to (14).

4 Steklov-Poincare's Averaging Technique in Obtaining Sequential Approximations

Our idea is based on the application of Steklov-Poincare's averaging technique to (14) in both space and time. This technique has been successfully applied in a number of applications (e.g., [23] and references therein). The technique will be applied for local averagings of the value function. The advantages of this approach is that the resulting model is simpler to solve compared to its classical HJB counterpart which contains the Hamiltonian defined as an infinum taken with respect to multifunction F from the differential inclusion of the dynamics (see, e.g., (1)–(3)).

In the space-time region

$$\Omega_0 = \{(x', t') : \ X_1 \leq x' \leq X_2, \ t \leq t' \leq t + \Delta t\} \tag{15}$$

where $X_1 = \min_{\theta_2 \in [-1,1]} (x, x + \theta_2 \Delta x)$ and $X_2 = \max_{\theta_2 \in [-1,1]} (x, x + \theta_2 \Delta x)$, we introduce the Steklov-Poincare averaging operators as follows

$$\psi_1(t, x) = S^t Q^u(t, x) = \int_t^{t+\theta_1 \Delta t} Q^u(x, \eta) d\eta, \quad 0 \leq \theta_1 \leq 1, \tag{16}$$

$$\psi_2(t, x) = S^x Q^u(t, x) = \int_x^{x+\theta_2|_{\{\theta_1=1\}}\Delta x} Q^u(\mu, t) d\eta, \quad -1 \leq \theta_2 \leq 1, \tag{17}$$

where t belongs to the control time interval $t \in [t_0, T]$, and in the general case $\theta_1 = \theta_1(x)$, $\theta_2 = \theta_2(t)$ such that $\forall \theta_1 \in [0, 1]$ and $\forall \theta_2 \in [-1, 1]$ we have that $[t, t + \theta_1 \Delta t] \times [x, x + \theta_2 \Delta x] \subseteq \Omega_0$. We denote $\omega_1 = \Delta x$ and $\omega_2 = \Delta t$, and make the only assumption about $Q^u(t, x)$ is that $Q^u(t, x) \in L^1(\Omega_0)$. Then for $\Delta t \to 0^+$ the ratio ω_1/ω_2 can be interpreted as the velocity of the controlled system in Ω_0 in a sense that in this space-time region we have $\lim_{\theta_1 \to 0^+} \theta_2 \Delta x / \theta_1 \Delta t = v$. By using the main property of Steklov-Poincare operators

$$\frac{\partial}{\partial t} \psi_1(t, x + \theta_2 \Delta x) = Q^u(t + \theta_1 \Delta t, x + \theta_2 \Delta x) - Q^u(t, x + \theta_2 \Delta x), \tag{18}$$

$$\frac{\partial}{\partial x} \psi_2(t + \theta_1 \Delta t, x) = Q^u(t + \theta_1 \Delta t, x + \theta_2 \Delta x) - Q^u(t + \theta_1 \Delta t, x), \tag{19}$$

we obtain the following identities

$$S^x S^t \frac{\partial Q^u(t, x + \theta_2 \Delta x)}{\partial t} = S^x Q^u(t + \theta_1 \Delta t, x + \theta_2 \Delta x) - S^x Q^u(t, x + \theta_2 \Delta x) \tag{20}$$

and

$$S^t S^x \frac{\partial Q^u(t + \theta_1 \Delta t, x)}{\partial x} = S^t Q^u(t + \theta_1 \Delta t, x + \theta_2 \Delta x) - S^t Q^u(t + \theta_1 \Delta t, x). \tag{21}$$

Therefore from (20) and (21) we have

$$\left[S^x \oplus S^t\right] Q^u(t+\theta_1\Delta t, x+\theta_2\Delta t) = S^x S^t \frac{\partial Q^u(t, x+\theta_2\Delta x)}{\partial t} +$$

$$S^t S^x \frac{\partial Q^u(t+\theta_1\Delta t, x)}{\partial x} + S^x Q^u(t, x+\theta_2\Delta x) + S^t Q^u(t+\theta_1\Delta t, x) \quad (22)$$

where $[S^x \oplus S^t]\, W \equiv S^x W + S^t W$. The last two terms in (22) are found as

$$S^x Q^u(t, x+\theta_2\Delta x) = S^x Q^u(t,x) + \theta_2\Delta x S^x \frac{\partial Q^u}{\partial x} +$$

$$\frac{1}{2} S^x \int_x^{x+\theta_2\Delta x} \frac{\partial^2 Q^u(\xi,t)}{\partial \xi^2}(x+\theta_2\Delta x-\xi)d\xi, \quad (23)$$

where $\xi \in [X_1, X_2]$, provided $Q^u \in L^1((t, t+\Delta t); W_1^1(X_1, X_2))$, and similarly

$$S^t Q^u(t+\theta_1\Delta t, x) = S^t Q^u(t,x) + \theta_1\Delta t S^t \frac{\partial Q^u}{\partial t} +$$

$$\frac{1}{2} S^t \int_t^{t+\theta_1\Delta t} \frac{\partial^2 Q^u(x,\eta)}{\partial \eta^2}(t+\theta_1\Delta t-\eta)d\eta, \quad (24)$$

where $\eta \in [t, t+\theta_1\Delta t]$, provided $Q^u \in W_1^{1,1}(\Omega_0)$.

Relationships (23) and (24) are substituted into (22) to get

$$[S^x \oplus S^t] Q^u(t+\theta_1\Delta t, x+\theta_2\Delta x) = [S^x \oplus S^t] Q^u(t,x) +$$

$$\left[\theta_2\Delta x S^x \frac{\partial Q^u}{\partial x} + \theta_1\Delta t S^t \frac{\partial Q^u}{\partial t}\right] +$$

$$\left[S^x S^t \frac{\partial Q^u(t, x+\theta_2\Delta x)}{\partial t} + S^t S^x \frac{\partial Q^u(t+\theta_1\Delta t, x)}{\partial x}\right] +$$

$$\left[\frac{1}{2} S^x \int_x^{x+\theta_2\Delta x} \frac{\partial^2 Q^u(\xi,t)}{\partial \xi^2}(x+\theta_2\Delta x-\xi)d\xi + \right.$$

$$\left. \frac{1}{2} S^t \int_t^{t+\theta_1\Delta t} \frac{\partial^2 Q^u(x,\eta)}{\partial \eta^2}(t+\theta_1\Delta t-\eta)d\eta\right]. \quad (25)$$

Since $Q^u \in W_1^{1,1}(\Omega_0)$ we transform the terms in square brackets of (25) by taking the Taylor's remainder in the integral form and grouping together

corresponding terms. This leads to

$$
\begin{aligned}
&[S^x \oplus S^t]Q^u(t+\theta_1\Delta t, x+\theta_2\Delta x) = [S^x \oplus S^t]Q^u(t,x) + \\
&\left[\theta_2\Delta x S^x \frac{\partial Q^u}{\partial x} + \theta_1 \Delta t S^t \frac{\partial Q^u}{\partial t}\right] + \left[S^x S^t \frac{\partial Q^u}{\partial t} + S^t S^x \frac{\partial Q^u}{\partial x}\right] + \\
&\left[\theta_2\Delta x S^x S^t \frac{\partial^2 Q^u}{\partial t \partial x} + \theta_1 \Delta t S^t S^x \frac{\partial^2 Q^u}{\partial x \partial t}\right] + \\
&\frac{1}{2}\left\{S^x S^t \frac{\partial}{\partial t}\int_x^{x+\theta_2\Delta x} \frac{\partial^2 Q^u(\xi,t)}{\partial \xi^2}(x+\theta_2\Delta x-\xi)d\xi + \right. \\
&\left. S^t S^x \frac{\partial}{\partial x}\int_t^{t+\theta_1\Delta t} \frac{\partial^2 Q^u(x,\eta)}{\partial \eta^2}(t+\theta_1\Delta t-\eta)d\eta\right\} + \\
&\frac{1}{2}\left\{S^x \frac{\partial}{\partial t}\int_x^{x+\theta_2\Delta x} \frac{\partial^2 Q^u(\xi,t)}{\partial \xi^2}(x+\theta_2\Delta x-\xi)d\xi + \right. \\
&\left. S^t \frac{\partial}{\partial x}\int_t^{t+\theta_1\Delta t} \frac{\partial^2 Q^u(x,\eta)}{\partial \eta^2}(t+\theta_1\Delta t-\eta)d\eta\right\}. \qquad (26)
\end{aligned}
$$

Further, we note that $\forall\theta_1 \in [0,1]$ and $\forall\theta_2 \in [-1,1]$

$$
\begin{aligned}
[S^x \oplus S^t]Q^u(t,x) \;=\; &\min_{(t',x')\in\Omega_0}\left\{(S^x \oplus S^t)\int_t^{t+\theta_1\Delta t} f_0 d\tau + \right. \\
&\left. (S^x \oplus S^t)Q^u(t+\theta_1\Delta t, x+\theta_2\Delta x)\right\}. \qquad (27)
\end{aligned}
$$

By using the regularisation theorem for optimal control algorithms (see [19], Theorem 4.1) it can be shown that there exists such a value of θ_1 (coupled to the behaviour of θ_2 that the dependency of f_0 on u can be replaced for its dependency on the performance measure Q^u. In brief, the following line of arguments can be applied. If we take the performance measure $Q(x,t;u(\tau), t \le \tau \le T) = \int_t^T f_0(x(\tilde\tau), u(\tau), \tilde\tau)d\tilde\tau$ and consider a class of functions represented in the form $Q' = \int_t^T f_0(x(\tilde\tau), Q, \tilde\tau)d\tilde\tau$, then, provided f_0 is Lipschitz with constant q in $u(\tau)$, we have $\int_{X_1}^{X_2}|Q-Q'|dx \le q\int_t^T\int_{X_1}^{X_2}|Q-u|dtdx$. Choosing Q from inequality $q\int\int|Q-u|dtdx < \epsilon$, where ϵ is an arbitrary small positive number, leads to a situation where Q and Q' become non-distinguishable in L^1. To put it differently, the description of the process in terms of (x,u) and (t,Q^u) are complementary. Hence, the original control problem can be reformulated in terms of function Q^u with

the following integro-differential equation (here and further in the paper the index u near Q^u is omitted)

$$\left[\theta_2 \Delta x S^x \frac{\partial Q}{\partial x} + \theta_1 \Delta t S^t \frac{\partial Q}{\partial t}\right] + \left[S^x S^t \frac{\partial Q}{\partial t} + S^t S^x \frac{\partial Q}{\partial x}\right] +$$

$$\left[\theta_2 \Delta x S^x S^t \frac{\partial^2 Q}{\partial t \partial x} + \theta_1 \Delta t S^t S^x \frac{\partial^2 Q}{\partial x \partial t}\right] +$$

$$\frac{1}{2}\left\{S^x S^t \frac{\partial}{\partial t}\int_x^{x+\theta_2\Delta x} \frac{\partial^2 Q(\xi,t)}{\partial \xi^2}(x+\theta_2\Delta x-\xi)d\xi + \right.$$

$$\left. S^t S^x \frac{\partial}{\partial x}\int_t^{t+\theta_1\Delta t} \frac{\partial^2 Q(x,\eta)}{\partial \eta^2}(t+\theta_1\Delta t-\eta)d\eta\right\} +$$

$$\frac{1}{2}\left\{S^x \frac{\partial}{\partial t}\int_x^{x+\theta_2\Delta x} \frac{\partial^2 Q(\xi,t)}{\partial \xi^2}(x+\theta_2\Delta x-\xi)d\xi + \right.$$

$$\left. S^t \frac{\partial}{\partial x}\int_t^{t+\theta_1\Delta t} \frac{\partial^2 Q(x,\eta)}{\partial \eta^2}(t+\theta_1\Delta t-\eta)d\eta\right\} +$$

$$(S^x \oplus S^t)\int_t^{t+\theta_1\Delta t} f_0 d\tau = 0. \tag{28}$$

For $Q \in W_1^{2,2}(\Omega_0)$ we transform the integro-differential equation to the following form

$$\frac{\partial Q}{\partial x}\left[\frac{\omega_2}{\omega_1}+1\right] + \frac{\partial Q}{\partial t}\left[\frac{\omega_1}{\omega_2}+1\right] + \omega_1 \frac{\partial^2 Q}{\partial x \partial t} + \omega_2 \frac{\partial^2 Q}{\partial t \partial x} + f_0\left(\frac{\omega_1}{\omega_2}+1\right) +$$

$$\frac{1}{2}\left\{\frac{1}{\omega_1}\int_x^{x+\theta_2\Delta x} \frac{\partial^2 Q(\xi,t+\theta_1\Delta t)}{\partial \xi^2}(x+\theta_2\Delta x-\xi)d\xi + \right.$$

$$\left. \frac{1}{\omega_2}\int_t^{t+\theta_1\Delta x} \frac{\partial^2 Q(x+\theta_2\Delta x,\eta)}{\partial \eta^2}(t+\theta_1\Delta t-\eta)d\eta\right\} = 0. \tag{29}$$

Finally, the last two terms can be transformed by recalling that if for an integrable function defined in Q_T, that is if $\mathcal{W}(x,t) \in L^1(Q_T)$, and $\int\int_{Q_T} |\mathcal{W}(\xi,\eta)| d\xi d\eta = 0$, then $\mathcal{W}(x,t) = 0$ a.e. in Q_T. Indeed, consider the integrals

$$\frac{1}{\omega_1}\int_x^{x+\theta_2\Delta x} \frac{\partial^2 Q(\xi,t+\theta_1\Delta t)}{\partial \xi^2}(x+\theta_2\Delta x-\xi)d\xi \tag{30}$$

and

$$\frac{1}{\omega_2}\int_t^{t+\theta_1\Delta t}\frac{\partial^2 Q(x+\theta_2\Delta x,\eta)}{\partial\eta^2}(t+\theta_1\Delta t-\eta)d\eta \tag{31}$$

for $t_0 \le t \le T$ and $\min_t x(t) \le x \le \max_t x(t)$, i.e. in all space-time region Q_T. First, we split this region into intervals $[x_{i-1}, x_{i-1}+\Delta\tilde{x}]$, $i = 1,...,N$ and $[t_{j-1}, t_{j-1}+\Delta\tilde{t}]$, $j = 1,...,M$. Then, we consider integrals (30) on these intervals and sum them up all over the whole region Q_T. The points ξ and η play the role of mathematical expectations that "random" variables x' and t' will belong to the intervals $[x_{i-1}, x_{i-1}+\Delta\tilde{x}]$ and $[t_{j-1}, t_{j-1}+\Delta\tilde{t}]$, respectively. Since

$$\int_x^{x+\theta_2\Delta x}\frac{\partial^2 Q(\xi,t+\theta_1\Delta t)}{\partial\xi^2}(x+\theta_2\Delta x-\xi)d\xi =$$

$$-\frac{1}{2}\int_x^{x+\theta_2\Delta x}\frac{\partial^2 Q(\xi,t+\theta_1\Delta t)}{\partial\xi^2}d(x+\theta_2\Delta x-\xi)^2, \tag{32}$$

we introduce the quantity $\sigma_1 = 1/(4\omega_1)\int_x^{x+\theta_2\Delta x} d(x+\theta_1\Delta x-\xi)^2$, $\xi \in [x, x+\theta_2\Delta x]$, which characterises the dispersion properties of the controlled process $x' \in [x, x+\theta_2\Delta x]$ with respect to t', or, in other words, θ_2 w.r.t. θ_1 whenever $\theta_1 \to 0^+$. In a sense, it gives a probabilistic measure of the location of a random point ξ in the interval $[x, x+\theta_2\Delta x]$ at time $t+\theta_1\Delta t$. Similarly, the quantity σ_2, defined as $\sigma_2 = 1/(4\omega_2)\int_t^{t+\theta_1\Delta t} d(t+\theta_1\Delta t-\eta)^2$, $\eta \in [t, t+\theta_1\Delta t]$, characterises θ_1 w.r.t. θ_2 whenever $\theta_2 \to 0$ and it gives a probabilistic measure of the location of a random point η in the time interval $[t, t+\theta_1\Delta t]$ for state $x+\theta_2\Delta x$. Finally, if we recall that $v \equiv f(x,t,Q) = \lim_{\theta_1\to 0}(\theta_2\Delta x)/(\theta_1\Delta t)$, we obtain the following partial differential equation:

$$(1+v)\left[\frac{\partial Q}{\partial x}+\frac{1}{v}\left(\frac{\partial Q}{\partial t}+f_0\right)\right]+\omega_1\frac{\partial^2 Q}{\partial x\partial t}+\omega_2\frac{\partial^2 Q}{\partial t\partial x}=\sigma_1\frac{\partial^2 Q}{\partial x^2}+\sigma_2\frac{\partial^2 Q}{\partial t^2}. \tag{33}$$

This representation is obtained for control problems under the assumption that $Q \in W_1^{2,2}(\Omega_0)$. This assumption can be relaxed. Indeed, let $\chi_{\Omega_0}(x,t)$ be the characteristic function of the region Ω_0 (i.e. $\chi_{\Omega_0}(x,t) = 1$ if $(x,t) \in \Omega_0$, and zero, otherwise). Then, function $Q(t,x) \in W_1^{1,1}(\Omega_0)$ is, by definition, the generalized solution of the control problem in Ω_0 if the integral identity

$$\int\int_{\Omega_0}\left\{S^x\otimes S^t\left[(1+v)\left(\frac{\partial Q}{\partial x}+\frac{1}{v}\left(\frac{\partial Q}{\partial t}+f_0\right)\right)+\right.\right.$$

$$\left.\left.\omega_1\frac{\partial^2 Q}{\partial t\partial x}+\omega_2\frac{\partial^2 Q}{\partial x\partial t}-\sigma_1\frac{\partial^2 Q}{\partial x^2}-\sigma_2\frac{\partial^2 Q}{\partial t^2}\right]\right\}\chi_{\Omega_0}(x,t)dxdt = 0 \tag{34}$$

is satisfied.

If $F \equiv f_0(1+1/v) \in L^1_{\mathrm{Lip}}(\Omega_0)$, that is $\|F(t,x,Q') - F(t,x,Q'')\|_{L^1(\Omega_0)} \leq q\|Q' - Q''\|_{L^1(\Omega_0)}$, and $f_0, g, f \in L^1(\Omega_0)$, then there exists a unique generalized solution of equation (33) which has mixed and second order derivatives as soon as $Q(t,x) \in W_1^{2,2}(\Omega_0)$. Other results along this line obtained for nonsmooth and purely deterministic approximations of (33) were first reported in [21]. In the later case the proof of the result analogous to that reported here is based on the Fejer's sums methodology.

The current development of the presented work, an intrinsic feature of which has been the consideration of nonsmooth deterministic and stochastic models with control functions dependent on both time and space variables, is carried out to characterise the local vector fields of the approximations discussed here. Such local vector fields are defined in terms of functions related to ψ_1^u and ψ_2^u, a pair of functions participating in the formulation of modified canonic equations. This development is based essentially on two major tools, the Steklov-Poincare operators, discussed in this work, and variational Lie derivatives.

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Summary: "Deterministic and stochastic dynamic problems in control theory can be reduced to partial differential equations by using Bellman's approach. The resulting PDE-based mathematical models are difficult to solve numerically, and most existing approaches cannot provide the accuracy expected in practical applications. We propose approximating the original controlled dynamics with a sequence of PDEs. By applying the Steklov-Poincaré operator technique the general form of equations in such a sequence is established. The model derived can be used in cases that are not covered by standard diffusion processes."