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A NOTE ON THE CLASS OF WEAKLY COUPLED PROBLEMS OF NON-STATIONARY PIEZOELECTRICITY

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SUMMARY

In this paper we deal with a model of coupled non-stationary electroelasticity with displacements and potential given on the boundary of a body. We construct a numerical scheme for modelling electromechanical interactions in the body, and present a spectrum of a priori estimates for the solution of this scheme. Such estimates allow us to prove the convergence of the scheme to a generalized solution of the differential problem from $W_2^2(Q_T)$ with the second order of accuracy in a weaker than L^2 metric.

KEY WORDS coupled models of dynamic electroelasticity; generalized solutions; metrics weaker than L^2; spectrum of a priori estimates

1. INTRODUCTION

Since the pioneering work of Pierre and Jacques Curie in 1880–1881, the piezoelectric effect has attracted a lot of attention from physicists, engineers, mathematicians and more recently biologists. In deformed piezoelectric materials we observe the presence of an electric field, and conversely these materials undergo deformation when subjected to an electric field. New technological advances during recent years have once again revitalized the significance of this effect in applications of intelligent structures[1] and biopolymers.[2]

Both new[3] and well-known applications[4] require rigorous mathematical approaches in the treatment of the intrinsic coupling phenomenon between electric and mechanical fields in piezoelectric materials.

The present paper complements earlier published papers[1,5,6] (see also references therein) where mathematical models and numerical procedures were investigated for the strongly coupled problems of dynamic electroelasticity. In this paper we investigate the situation where weak coupling occurs in long hollow piezoceramic cylinders. Such cylinders are widely used in many different applications due to their ability to accept much larger external mechanical loadings compared to planar elements. Piezoelectric cylinders may also provide a good model for studying

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electromechanical processes in bone tissues. In a number of applications that involve hollow piezoelectric cylinders it is important to consider coupled electromechanical fields that have a non-stationary, rather than a steady-state, character. This complicates the mathematical investigation of associated models.

We have organized this paper as follows. Section 2 provides the reader with the mathematical model of the problem and discusses difficulties arising as a result of coupling. In Section 3 we present an operator formulation of the problem. We show that the solution of non-stationary coupled problems of electroelasticity with given displacements on the boundary is equivalent to the solution of a hyperbolic equation with a positive definite self-adjoint spatial operator. Section 4 gives results of the construction and analysis of a numerical scheme for the solution of the problem. We prove that the solution of the numerical scheme converges to the solution of the original differential problem from the class $W_2^2(Q_T)$ with second-order accuracy. Conclusions and future directions are discussed in Section 5.

2. GOVERNING EQUATIONS AND THEIR COUPLING

Mathematically a coupled phenomenon of electromechanical interaction in hollow piezoelectric cylinders can be appropriately described by the system of partial differential equations[5,6] which includes the equation of motion for continuum medium

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r) - \frac{\sigma\Theta}{r} + f_1(r, t) \tag{1}$$

and the Maxwell equation for a piezoelectric medium in the acoustic range of frequencies, i.e. the equation of forced electrostatics for dielectrics

$$\frac{1}{r}\frac{\partial}{\partial r}(rD_r) = f_2(r, t) \tag{2}$$

The system (1)–(2) is considered in the space–time region $Q_T = \{(x,t)\colon R_0 < x < R_1, 0 < t < T\}$. The notation used in this paper is the same as in Reference 6, namely:

u = function of radial displacements
φ = electric potential
E_r = radial component of electric field strength
D_r = electrical component of electric induction
σ_r and σ_Θ = components of field of stresses
e_{ij} = given constants of piezomoduli
c_{kl} = given constants of elastic moduli
ε_{11} = given dielectric permittivity
ρ = given density of material
f_1 = given function for density of mass forces within body
f_2 = given function for electric charge density within body.

We emphasize that the electric variables in (1)–(2) are not static because of the coupling to the dynamic 'mechanical' equation (1). There are two physically and mathematically distinct cases of coupling of these equations that are typical in applications. Namely, that of radial preliminary polarization, and that of circumferential preliminary polarization (see Reference 7 for details

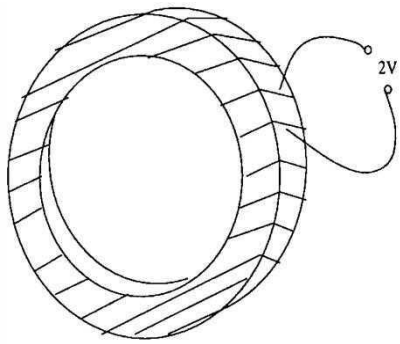

Figure 1. The case of circumferential preliminary polarization

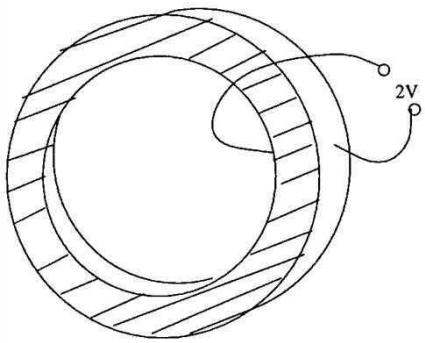

Figure 2. The case of radial preliminary polarization

of the latter case). From the physical point of view these situations are well elucidated in the literature and we only provide figures that correspond to different types of preliminary polarization (see Figures 1 and 2).

The cases are distinguished by the fact that for circumferential preliminary polarization a practically uncoupled system is obtained provided displacements on boundaries are known, whereas for radial preliminary polarization there is a strong coupling between electrical and mechanical fields.[5,6] In the latter case equations (1) and (2) are coupled by the following state equations:

$$\begin{cases} \sigma_r = c_{11}\varepsilon_r + c_{12}\varepsilon_\Theta - e_{11}E_r \\ \sigma_\Theta = c_{12}\varepsilon_r + c_{22}\varepsilon_\Theta - e_{12}E_r \\ D_r = \varepsilon_{11}E_r + e_{12}\varepsilon_\Theta + e_{11}\varepsilon_r \end{cases} \tag{3}$$

where the Cauchy relations between components of the field stresses, σ_r, σ_Θ, and components of the field of deformations, ε_r, ε_Θ, as well as the formula for electrostatic potential φ are given as follows:

$$\varepsilon_r = \frac{\partial u}{\partial r}, \varepsilon_\Theta = \frac{u}{r}, \ E_r = -\frac{\partial \varphi}{\partial r} \tag{4}$$

The model (1)–(4) is supplemented by the initial conditions given for the displacement and its rates of change,

$$u(r,0) = u_0(r), \frac{\partial u(r,0)}{\partial t} = u_1(r) \tag{5}$$

As usual we assume non-negativeness of the potential energy of deformation.[5] To complete the formulation of the model we have to adequately formulate boundary conditions.

Due to the intrinsic complexity of finite domain problems, mathematical models in coupled field theory are often simplified by a consideration of infinite domain problems for which it is sometimes possible to develop a technique for obtaining the analytical closed-form solutions. Of course, in some special cases such analytical solutions for infinite domain problems may help to improve the accuracy and efficiency of numerical techniques based on finite element, finite boundary, finite difference methods or their combinations. However, in many other practically important cases such a simplification may be inappropriate.

On the other hand, the mathematical model has to be formulated with the aim of reflecting a compromise between the *complexity of the coupling phenomenon* and the *efficiency of a practical solution* to the problem. In References 5 and 6 (and references therein) the problem (1)–(5) was studied extensively. The coupling of the equations (1) and (2) was amplified not only by the state equations (3), but also by the boundary conditions for stresses. This paper deals with a somewhat different situation where it is appropriate to assume that displacements are given on the boundaries of the body. Such an assumption, typical in a number of applications, leads to improved accuracy in the results obtained in earlier papers.

3. OPERATOR FORMULATION OF THE WEAKLY COUPLED MODEL OF DYNAMIC ELECTROELASTICITY

It is typical in a number of applications that displacements on the boundary of a piezoelectric body are known at least approximately. In such situations the equation of motion and the equation of forced electrostatics for dielectrics are coupled *only* by the state equations. This provides an essential simplification of the mathematical analysis of the problem compared with applications where coupling through boundary conditions for stresses is required.[5,6]

Let us consider (without loss of generality) the problem (1)–(5) for non-homogeneous boundary conditions for the potential and homogeneous boundary conditions for displacements (instead of the definition of stresses on internal and external surfaces of a body). For a function ω, which could represent either u or φ, we introduce the following operators:

$$\mathcal{L}_1(\omega) = \frac{1}{r}\frac{\partial}{\partial r}(r\omega),\ \mathcal{L}_2(\omega) = \frac{1}{r}\omega,\ \mathcal{R}(\omega) = \frac{\partial \omega}{\partial r} \tag{6}$$

and denote

$$\mathcal{A}_1 = -c_{11}\mathcal{L}_1 \cdot \mathcal{R} + c_{22}\mathcal{L}_2^2,\ \mathcal{C}_1 = -e_{11}\mathcal{L}_1 \cdot \mathcal{R} + e_{12}\mathcal{L}_2 \cdot \mathcal{R} \tag{7}$$

$$\mathcal{A}_2 = -\varepsilon_{11}\mathcal{L}_1 \cdot \mathcal{R},\ \mathcal{C}_2 = e_{11}\mathcal{L}_1 \cdot \mathcal{R} + e_{12}\mathcal{L}_1 \cdot \mathcal{L}_2,\ \partial G = \{R_0,\ R_1\} \tag{8}$$

Then using (6)–(8) the mathematical model for the description of electromechanical interactions in a piezoelectric body can be represented as follows:

$$\rho \frac{\partial^2 u}{\partial t^2} + \mathcal{A}_1 u + \mathcal{C}_1 \varphi = f_1 \tag{9}$$

$$\mathcal{A}_2 \varphi + \mathcal{C}_2 u = f_2 \tag{10}$$

with boundary and initial conditions respectively

$$u|_{\partial G} = 0, \ \varphi|_{\partial G} = \pm V(t) \tag{11}$$

$$u|_{t=0} = u_0, \ \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1 \tag{12}$$

where $V(t)$ denotes the absolute value of the given potential function on the internal and external surfaces of the cylinder (see Figure 2). The model (9)–(12) is a coupled system of dynamic electroelasticity for which it is possible to obtain a stronger result than the results obtained in References 5 and 6 for the coupled system of electroelasticity with the stresses given on the boundary.

From equation (10) we find explicitly the value of the potential and substitute it into equation (9) to get

$$\rho \frac{\partial^2 u}{\partial t^2} + \mathcal{A} u = \mathcal{F} \tag{13}$$

where

$$\mathcal{A} = \mathcal{A}_1 - \mathcal{C}_1 \cdot \mathcal{A}_2^{-1} \cdot \mathcal{C}_2, \ \mathcal{F} = f_1 - \mathcal{C}_1 \cdot \mathcal{A}_2^{-1} f_2 \tag{14}$$

Using (6)–(8) it is straightforward to show that

$$\mathcal{A} = -\left(c_{11} + \frac{e_{11}^2}{\varepsilon_{11}}\right)\mathcal{L}_1 \cdot \mathcal{R} + \left(c_{22} + \frac{e_{12}^2}{\varepsilon_{11}}\right)\mathcal{L}_2^2 \tag{15}$$

Hence, the operator A is a positive-definite and self-adjoint operator.

4. NUMERICAL SCHEME, SPECTRUM OF A PRIORI ESTIMATES AND CONVERGENCE

Coupling elastic and electric fields, as well as anisotropy of physical properties of piezoelectric materials (for example, elastic, piezoelectric and dielectric moduli of piezoceramic) complicates the analysis of wave phenomena that take place in piezoelectric bodies compared to pure elastic materials. Even the one-dimensional problem in the non-stationary case presents a mathematically challenging and physically important problem. To obtain a plausible picture of the coupling phenomenon, methods for the solution of dynamical problems of electroelasticity may not always be based on averaging in thickness of mechanical components of electro-elastic fields and the subsequent use of the Kirchoff–Lave-type hypothesis. Such averaging techniques, successful in the theory of plates and shells applied to elastic bodies, may not be appropriate in coupled field

theory. Instead it is often more appropriate to apply directly an effective numerical technique to the original non-stationary coupled problem.

Let us introduce a difference grid covering the region Q_T as $\bar{\omega}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau$, where $\bar{\omega}_h = \{r_i = R_0 + ih,\ h = (R_1 - R_0)/N,\ i = 0,1,\ \ldots,\ N\}$, $\bar{\omega}_\tau = \{t_j = j\tau,\ \tau = T/L,\ j = 0,1,\ \ldots,\ L\}$, and let the functions y and μ be functions of two discrete variables defined on this grid which approximate the functions of displacement $u(r,t)$ and electrostatic potential $\varphi(r,t)$, respectively. Then, as we have shown in Section 3 for the differential case, we may obtain the following model for the numerical solution of (13), (14), (11), (12):

$$\mathcal{D}_1 y_{\bar{t}t} + \tilde{\mathcal{A}} y = \tilde{\mathcal{F}} \tag{16}$$

$$y\,|_{\partial G} = 0 \tag{17}$$

$$y\,|_{t=0} = u_0,\ \rho y_t\,|_{t=0} = \rho u_1 + \frac{\tau}{2}\tilde{\mathcal{A}} y$$

where

$$\tilde{\mathcal{A}} y = -\left(c_{11} + \frac{e_{11}^2}{\varepsilon_{11}}\right)\frac{1}{r}(\bar{r} y_{\bar{r}})_r + \left(c_{22} + \frac{e_{12}^2}{\varepsilon_{11}}\right)\frac{(y^{(+1)} + y)/(2\bar{r}^{(+1)}) + (y + y^{(-1)})/(2\bar{r})}{2r} \tag{19}$$

$$\mathcal{D}_1 y = \rho y,\ \tilde{\mathcal{C}}_1 y = -e_{11}\frac{1}{r}(\bar{r} y_{\bar{r}})_r + e_{12}\frac{y_{\bar{r}}^{(+1)} + y_{\bar{r}}}{2r},\ \tilde{\mathcal{F}} = \varphi_1 - \tilde{\mathcal{C}}_1 \cdot \tilde{\mathcal{A}}_2^{-1}\varphi_2 \tag{20}$$

$$\tilde{\mathcal{A}}_2 y = -\varepsilon_{11}\frac{1}{r}(\bar{r} y_{\bar{r}})_r,\ \tilde{\mathcal{C}}_2 y = e_{11}\frac{1}{r}(\bar{r} y_{\bar{r}})_r + e_{12}\frac{1}{r}\left(\frac{y + y^{(-1)}}{2}\right)_r \tag{21}$$

All notation in (16)–(21) is taken from the theory of difference schemes,[8–10] where, for example,

by $\gamma_{\bar{t}t}$ we denote the second central difference derivative of the function γ in time, i.e. $\gamma_{\bar{t}t} = (\gamma^{(+1)} - 2\gamma + \gamma^{(-1)})/\tau^2$
by γ_r we denote the forward difference derivative of the function γ in space
by $\gamma_{\bar{r}}$ we denote the backward difference derivative of the function γ in space, etc.

Provided the problem (16)–(21) is solved we can determine the approximation to the potential as follows:

$$\mu = \tilde{\mathcal{A}}_2^{-1}\varphi_2 - \tilde{\mathcal{A}}_2^{-1} \cdot \tilde{\mathcal{C}}_2 y,\ \mu\,|_{\partial G} = \pm V \tag{22}$$

We assume that $y \equiv y(t) \in \mathrm{H}$, with H the Hilbert space of grid functions $y(t)$, $t \in \bar{\omega}_\tau$, and the scalar product denoted by (y,v). Let us further denote

$$\|y(t)\|_*^2 = \|y(t)\|_{\bar{\mathcal{A}}^{-1}}^2 + \left\| \sum_{t'=0}^{t} \tau y(t') \right\|^2 \tag{23}$$

$$\|y\|_0^2 = \sum_{t=0}^{T} \tau \|y(t)\|^2,\ \bar{\mathcal{A}} = \tilde{\mathcal{A}}^{-1} \cdot (\mathcal{D}_1 + \tau^2 \tilde{\mathcal{A}}/2),\ \bar{\mathcal{B}} = \tilde{\mathcal{A}}^{-1} \cdot \mathcal{D}_1 \tag{24}$$

where, as usual,

$$\|y(t)\|^2 = (y(t), y(t)), \;\; \|y(t)\|^2_{\tilde{\mathcal{A}}} = (\tilde{\mathcal{A}}y(t), y(t))$$

Using the technique developed for difference analogues of hyperbolic equations (see References 8 and 11 and references therein), we come to the following result.

Theorem 1. If the stability condition

$$\mathcal{D}_1 \geqslant (1+\varepsilon)\tau^2\tilde{\mathcal{A}}/4, \;\; \varepsilon > 0 \tag{25}$$

is satisfied, then the solution of the problem (16)–(22) is characterized by the following spectrum of a priori estimates:

(i) in the grid analogue of the norm L^2,

$$\|y(t)\| \leqslant M_1\left(\|y(0)\|_{\mathcal{D}_1} + \|\mathcal{D}_1 y_t(0)\|_{\tilde{\mathcal{A}}^{-1}} + \sum_{s=1}^{n}\|\tilde{\mathcal{F}}_s\|_{\tilde{\mathcal{A}}^{-1}}\right) \tag{26}$$

$$\|\mu(t)\| \leqslant M_2\left(\|y(0)\|_{\mathcal{D}_1} + \|\mathcal{D}_1 y_{\mathrm{t}}(0)\|_{\tilde{\mathcal{A}}^{-1}} + \sum_{s=1}^{n}\|\tilde{\mathcal{F}}_s\|_{\tilde{\mathcal{A}}^{-1}} + \|\mathcal{A}_2^{-1}\varphi_2\|\right) \tag{27}$$

(ii) in the grid analogue of the energy norm,

$$\|y(t)\|_{\tilde{\mathcal{A}}} \leqslant M_1\left(\|y(0)\|_{\tilde{\mathcal{A}}} + \|y_t(0)\|_{\mathcal{D}_1} + \sum_{s=1}^{n}\|\tilde{\mathcal{F}}_s\|_{\mathcal{D}^{-1}}\right) \tag{28}$$

$$\|\mu(t)\|_{\tilde{\mathcal{A}}} \leqslant M_2\left(\|y(0)\|_{\tilde{\mathcal{A}}} + \|y_{\mathrm{t}}(0)\|_{\mathcal{D}_1} + \sum_{s=1}^{n}\|\tilde{\mathcal{F}}_s\|_{\mathcal{D}^{-1}} + \|\tilde{\mathcal{A}}_2^{-1}\varphi_2\|_{\tilde{\mathcal{A}}}\right) \tag{29}$$

(iii) in the star-norm which is weaker than the grid analogue of the norm L^2,

$$\|y(t)\|_* \leqslant M_1\left(\|y(0)\|_{\tilde{\mathcal{A}}} + \|\bar{\mathcal{B}}y_t(0)\| + \|\tilde{\mathcal{A}}\tilde{\mathcal{F}}\|_0\right) \tag{30}$$

$$\|\mu(t)\|_{\tilde{\mathcal{A}}^{-1}} \leqslant M_2(\|y(0)\|_{\tilde{\mathcal{A}}} + \|\bar{\mathcal{B}}y_{\mathrm{t}}(0)\| + \|\tilde{\mathcal{A}}\tilde{\mathcal{F}}\|_0 + \|\tilde{\mathcal{A}}_2^{-1}\varphi_2\|_{\tilde{\mathcal{A}}^{-1}}) \tag{31}$$

Proof. Estimates (26)–(29) follow from a priori estimates obtained in Reference 8. The proof of (30) is based on Theorem 1 from Reference 11. Finally, estimate (31) follows from relationship (22).

Theorem 1 provides a way to prove the convergence of the numerical scheme (16)–(22) to the generalized solution of the non-stationary problem of dynamic electroelasticity (9)–(12).

Corollary 1. The numerical scheme (16)–(22) converges to the generalized solution of the problem (9)–(12) from the class $W_2^2(Q_T)$ with the second order of accuracy in the star-norm (23).

The proof of Corollary 1 follows immediately from the application of the technique described in References 8 and 11 and our a priori estimates (30), (31). The existence and uniqueness theory

for coupled dynamic electroelasticity in classes of generalized solutions was developed in Reference 12.

We emphasize that the norm defined by (23) is weaker than the conventional grid norm L^2. However, having a spectrum of a priori estimates in different functional spaces and using the Banach-space-interpolation technique,[11] we can obtain the rate of convergence in those functional spaces that is most suitable for specific applications.

Remark 1. Results of computational experiments on the investigation of coupled electromechanical interactions in the non-stationary case can be found in previous papers of this sequence (see References 5 and 6 and references therein).

5. CONCLUDING REMARKS AND FUTURE DIRECTIONS

In this paper we have presented results on analysis of numerical schemes applied to the important practical case of coupled non-stationary electroelasticity. Under minimal requirements on solution smoothness we have proved that our numerical scheme has second-order accuracy in a metric weaker than L^2. Such metrics are essential in the analysis of numerical discretizations of hyperbolic type equations and in the analysis of mathematical models of coupled field theory that contain a hyperbolic mode.

As a future direction of this work we mention the extension of the presented results to a wider class of problems in coupled field theory. Along this direction are also the problems of thermo-electroelasticity and opto-piezoelectricity.

During recent years piezoelectrics has become effective in a wide range of applications as an integral part of smart materials and structures (for example, as sensors and/or actuators). In many such applications temperature fluctuations may substantially influence the overall performance of structures. When the electromechanical coupling is large the temperature coefficients of all constants (piezoelectric, dielectric and elastic) may be significant. Even for materials with small electromechanical coupling temperature coefficients of the elastic constants may be substantial. In such cases it is important to consider thermal, electric and elastic fields as a unified whole, formulating the mathematical model as a problem of thermo-electroelasticity.

A number of technical devices, including actuators, have to work under exposure to high-energy lights (such as lasers). In the design of such devices knowledge of the opto-thermo-electromechanical behaviour of the material is the key element in a successful design. Such practical applications lead to new challenges in the mathematical study of opto-piezoelectricity.

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