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Dynamic System Evolution and Markov Chain Approximation

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In this paper computational aspects of the mathematical modelling of dynamic system evolution have been considered as a problem in information theory. The construction of mathematical models is treated as a decision making process with limited available information. The solution of the problem is associated with a computational model based on heuristics of a Markov Chain in a discrete space–time of events. A stable approximation of the chain has been derived and the limiting cases are discussed. An intrinsic interconnection of constructive, sequential, and evolutionary approaches in related optimization problems provides new challenges for future work.

Keywords: Decision making with limited information, Optimal control theory, Hyperbolicity of dynamic rules, Generalized dynamic systems, Markov Chain approximation

1 INTRODUCTION

Many mathematical problems in information theory and optimal control related to dynamic system studies can be formulated in the following generic form. A decision maker (DM, i.e. problem solver, modeler or observer) receives information about a system from observations, measurements, or computations in the form of a data stream that can be formalized mathematically as a *sequence*

$$(x_0, x_1, \ldots). \tag{1.1}$$

We assume that such a sequence has at least *two elements* and that each element of the sequence is labelled by its own *time* t. Hence, referring to the element x_t of the sequence, we assume that the total amount of information about the system that corresponds to the time interval $(0, t)$ of its behaviour has been received, or at least can be received in principle. Under the above assumptions we can introduce a *set* T_t of permissible strategies for each time t. Then, observing the sequence $(x_0, \ldots, x_t)$, the decision maker can *choose* a strategy that is defined by the inclusion

$$s_t \in \bigcup_{\tau=0}^{t} T_\tau. \tag{1.2}$$

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Typically we reduce the problem of constructing a map between elements x_t and s_t defined by (1.1), (1.2) to a simpler problem allowing the set of permissible strategies for all times of consideration to be *fixed* and to be given *a priori*. Namely, we can idealize *actions* of the decision maker as follows. We can assume that the DM *can* select a strategy s_t at each time t from a given set U_T. Of course, the validity of such a simplification ultimately depends on the Axiom of Choice excluding the logically possible case of incomparability of two arbitrary sets that correspond to two different times t and t' [51,32]. However, on the other hand, such a simplification permits the development of a set-theoretic approach to dynamic system evolution, and simplifies the mathematical formalizations of complex optimization problems. In fact, we can introduce a loss function $l(\cdot,\cdot)$ as a function of two variables, states x_t and strategies s_t, which are both characterized by the same time t. A desire to minimize time-averaging characteristics of this function can be formalized through the optimization problem

$$F(l) \to \min, \quad s_t \in U_T. \tag{1.3}$$

Here, the objective functional F may be, for example, the Cesaro-type sum

$$F(l) = \frac{1}{T}\sum_{k=0}^{n} l(x_{\tau_k}, s_{\tau_k}), \tag{1.4}$$

where $\tau_k \in (0,T)\ \forall k \in (0,1,\ldots,n)$ and T is assumed to be given. The limiting problem in the spirit of classic ergodic theorems arises when we investigate the limit behaviour

$$\lim_{T\to\infty} F(l)$$

with $F(l)$ given by (1.4). Objective criteria may also be formulated in an integral form. For example, for the Boltz problem in optimal control theory we have the form of the functional in (1.3)

$$F(l) = g(x_T) + \int_0^T f_0(\tau, x_\tau, s_\tau)\,\mathrm{d}\tau \quad x_T \in K, \tag{1.5}$$

where $T \in (0,\infty)$ is assumed to be given, and K is a given target set[a]. We can also consider a class of problems with infinite time horizon using discounting cost procedures. All these examples provide important partial cases of the general problem (1.1), (1.3).

Of course, to complete the formulation of the problem (1.1), (1.3) mathematically, we have to specify in what sense the sequence $\{x_t\}$ in (1.1) should be understood. One possible specification can be provided by an assumption that x_t may be appropriately described by a *given stationary ergodic distribution*. Then a typical assumption imposed on functions s_t from U_T is Lebesgue-measurability on the interval $(0,t)$. Under the above mentioned assumptions, associated theoretical issues are often addressed using the theory of Markov processes [19]. Starting from the work of Bellman [5,6], the theory has been extensively developed, and a number of efficient algorithms have been proposed. Discrete dynamic programming ideas have been essentially generalized for the continuous case during the past decades [18,19], and many new results that appeared recently indicate the continued research interest in these topics [19,35]. It should be noted, however, that many results in this area rely (explicitly or implicitly) on the assumption that a measurable function of strategies $s_t \in U_T$ may be effectively approximated using past states $x_{t'}$, $0 < t' < t$. If such an assumption is made, the attainability of the minimum in (1.3) becomes the subject of a corresponding smoothness assumption on the loss function [42]. On the other hand, regularity of this function is strongly dependent on *complete* information about the past states, and eventually on model data and parameters. Since the initial data for the model can only be known approximately, the whole stream of information available to the decision maker at time t can be interpreted, at best, as an approximation of system dynamics. The quality of such an approximation at time t is defined by the "informational completeness" of

[a] The functions f_0 and g are called running and terminal costs respectively. If $f_0 \equiv 0$ we have Mayer's problem whereas for $g \equiv 0$ the problem is referred to as the Lagrange problem.

the data stream

$$(s_0, x_0; s_1, x_1; \ldots s_{t'}, x_{t'}; \ldots) \tag{1.6}$$

when $t' \to t$. To complete the step corresponding to time t in this process, one can assume that the strategy s_t may be chosen from the same set U_T. Then, the next stream element x_t may be received with a given accuracy, at least in principle, if we also assume that element x_0 in (1.1) may be given with infinite precision. Of course, in the reality of mathematical modelling the latter assumption cannot be rigorously justified [45]. However, if strategies are chosen at each step to satisfy a certain subgoal, the described process provides the possibility of *evaluating* the quality of satisfaction of a subgoal that corresponds to time t. If the process is finite then we can refer to the last subgoal as a top-level goal [33]. The latter can be satisfied by satisfying subgoals at each step appealing to multicriteria analysis of the underlying problem.

The main problems in such analysis stem from the *coupling* of the sequence of subgoals to the definition of the top-level goal in the form of a functional of the loss function $l(\cdot,\cdot)$. Mathematically speaking, we should be able to define a mapping between fixed-time subgoal functions and an averaged-time goal functional. Such a definition is closely connected with the definition of optimal strategies which we do not know *a priori*. However, if it is known that $s_t \in U_T$, then it is reasonable to choose strategies based on knowledge not only of time t, but also on states x_t. If we assume further that x_t "accumulates" all past information about the system, then the concept of a Markov Chain comes by itself. Because of uncertainty in knowledge base (1.1), such an accumulation cannot be understood in a purely deterministic way [8]. The origin of such uncertainty is induced by the strategy s_0 in the data stream (1.6). However, mathematically such uncertainty can be formalized if instead of constraints (1.2) we consider "relaxed" constraints

$$s_t \in U_T, \tag{1.7}$$

assuming that the set U_T is given *a priori* for the whole time-set of interest. Then, instead of the data stream (1.6), we can consider an *informationally reduced stream*:

$$(x_0, (s_1, x_1); \ldots (s_{t'}, x_{t'}); \ldots), \tag{1.8}$$

where all strategies satisfy the constraints (1.7). An additional assumption of continuity of the sequence (1.1) in time allows a convenient mathematical framework for justification of models based on an approximation of (1.6) by (1.8). Such a classical idealization of temporal evolution by continuous trajectories of phase points, induced by classical mechanics, can be applied only within certain limited contexts, and involves serious difficulties in many areas of mathematical modelling. The main problems are caused by the fact that there are many dynamic systems for which arbitrary close initial conditions can give rise to qualitatively distinct (including exponentially diverging) types of trajectories [45]. Such strong trajectory instability requires other approaches in the description of dynamic system evolution. Under a probabilistic approach, deterministic invariance of phase points along trajectories is replaced by the invariance of the density along trajectories. *Physically*, such a "conservation of extension in phase" (due to J. Gibbs [37]) eventually requires a *construction* of Gibbs distribution functions using a probabilistic description of states. Mathematically speaking, this problem can be seen as a problem of a "closure" of the reduced informational stream (1.8) with respect to all possible states. Such a closure can be performed if we assume Lebesgue integrability of the function

$$\eta(\omega) = \begin{cases} -\omega \log \omega, & \omega > 0, \\ 0, & \omega = 0, \end{cases} \tag{1.9}$$

over the set Σ of all *possible* states, where $\omega = f(t, x_t)$ is the density function. From an information theory perspective, this logical step, which in the end requires answering the question of system stability, is equivalent to a transformation from the classic Shannon entropy [53,49] to

the Boltzmann–Gibbs entropy [37]. Under such a transformation we formally identify a (thermo)-dynamic system with a measure space [37]. If Σ is fixed and the measure is defined as a Lebesgue measure, then for any time-set $(0, T)$ (including the possibility of $T \to \infty$) the validity of the above transformation requires an *a priori* assumption of lower semi-continuity [55] of the recursive function

$$\zeta(\omega) = f_n(f_{n-1}(\ldots f_1(\omega)\ldots)) \tag{1.10}$$

as a function of density, where a theoretical possibility of $n \to \infty$ is permitted. If we assume that such a function exists, then in principle, the only possible uncertainty in the model (1.3), (1.8) for any $t = T$ is induced by the definition of x_0 and $\zeta(f(x, T))$. Such is indeed the case in optimal control theory where the recursive function ζ plays the role of the value function. In fact, if we know *a priori* that the top-level goal can be described appropriately by a continuous function $F(l)$, then the associated optimal control problems can be studied through a nonlinear backward evolution PDE known as the Hamilton–Jacobi–Bellman equation with Cauchy-type terminal conditions ([11,19] and references therin). If an algorithm for the numerical solution of the latter problem exists, it can in principle be represented in the form of the informational stream

$$((\zeta_{x_T}, s_T); (\zeta_{x_{T-\Delta t}}, s_{x_{T-\Delta t}}); \ldots (\zeta_{x_t}, s_{x_t}); \ldots), \tag{1.11}$$

when $t \to 0^+$ and $\Delta t > 0$. The main theoretical difficulty in the rigorous justification of algorithmic rules constructed according to (1.11) is the existence of the limit of s_{x_t} when $t \to 0^+$. If we assume that such a limit exists, then we should be able to evaluate the quantity

$$s_0 = \lim_{t \to 0^+} s_{x_t} \tag{1.12}$$

on the basis of ζ_{x_T} (which is assumed to be given) and some logical rules. In reality, the recursive function of density (1.10) at a fixed moment of time may be given only approximately. Such an approximation defines a *degree n* of the underlying recursion (1.10), and in turn defines a basic structure of a finite lattice on which the system dynamic can be approximated [14].

Hence, in general, information on an *approximation* of the same dynamic system can be provided in two possible ways:

- using the sequence (1.1), and
- using a subsequence of (1.11) that is $(\zeta_{x_T}, \zeta_{x_{T-\Delta t}}, \ldots)$.

Due to intrinsic uncertainty in the definitions of x_0 and ζ_{x_T}, neither of these approximations considered separately from the other can guarantee the adequacy of the approximation to the real system. However, we can draw certain conclusions on the system dynamics by analysing both of the sequences *simultaneously*. The complexity of such analysis is due to the necessity of a coupled investigation of the same system in two different scales. Mathematically, such scales are induced by the two limiting types of system behaviour with respect to the time-component: $t \to \infty$ and $\Delta t \to 0^+$. They are connected by the definition of the recursive degree for the system density, and ultimately, on the definition of the top-level goal in (1.3). Splitting up such a goal into subgoals provides an efficient method for the analysis of the system dynamics. In turn, such analysis gives a way to derive a sequential approximation of the system Hamiltonian, ensuring a stable model of system dynamics.

The remaining part of the paper is organized as follows. In Section 2 basic preliminaries are recalled for the formulation of optimal control problems as problems in information theory. Section 3 is devoted to consideration of deterministic and stochastic dynamic rules. Examples are given to show that if such rules are specified, then an informationally consistent formulation of control problems requires an analysis of system stability. Section 4 deals with deterministic and probabilistic algorithmic machines and analyses problems involved in their application. Section 5 gives a link between the questions discussed in the previous sections and discrete optimization problems using their common physical and informational basis. In Sections 6 and 7 mathematical

models are constructed and computational models derived to analyse dynamic system evolution using the Markov Chain approximations. A stable approximation for the hyperbolic model is obtained and the algorithm has been given. Computational aspects of Discrete Markov Decision Processes (DMDP) are discussed in Section 8. The main conclusions are summarized in Section 9.

2 PRELIMINARIES

Let us define the state space of the system by Σ and the Borel σ-algebra induced[b] by Σ as $\mathcal{B}(\Sigma)$. Then, no matter what the time-partition in $[0, t)$ is, $0 \le \tau_1 < \tau_2 < \cdots < \tau_n < \tau$, $\tau \in (0, t)$, we assume that $\forall X \in \mathcal{B}$:

$$P(x_\tau \in X | x_{\tau_1}, \ldots, x_{\tau_n}) = P(x_\tau \in X | x_{\tau_n}) \qquad (2.1)$$

almost surely[c]. That is, the data stream x_t *under the strategy of the time partition* has the Markovian property. Of course, continuity of the data stream x_t in t does not follow from the condition (2.1). Furthermore, even if x_t is a continuous function of time, it does not, on any account, mean that strategies form a continuous function of time as well. In general, we have a multicriteria optimization problem induced by the partition of time and the analysis of the sequence (1.6). However, the difficulty in evaluating the limit (1.12) prompts several ways to further simplify the problem. One of the direct ways is to assume *a priori* continuity of the sequence (1.1) in time. Then we can reformulate the multicriteria optimization problem arising in analysis of (1.6) as an optimal control problem (1.3) with respect to a continuous function of time $F(l)$ and some dynamic rules that define the sequence (1.1).

Alternatively, we can analyse the sequence (1.6) using DMDP. The theory of DMDP is well-developed under the assumption of the possibility of complete information in (1.6). During recent years new challenging problems have stimulated further development in the theory of DMDP [34,25,17]. In brief, one of the most interesting problems in this field is induced by the question of data perturbations in the informational stream (1.6). Indeed, when perturbations of a Markov Chain change its ergodic structure, the stationary distribution of the perturbed system may not be a continuous function [52,1]. Hence it is reasonable to assume that system dynamics depend on some parameters of the Markov Chain and due to the imprecision of available information we can study system dynamics using in general Singularly Perturbed Markov Chains (SPMC). In this framework evolution of a system is coupled to its Markov Chain parameters. An example of this type DMDP was provided in [13] where *non-diffusion stochastic models* were studied. We assume that in general the parameters of the Markov Chains are allowed to jump, and the jumping rates may be dependent on the state function x_t. The corresponding systems described by x_τ at time τ are called *piecewise-deterministic stochastic systems*. Such systems have been extensively studied during recent time by theoretical physicists [29], and indicate growing interest in hyperbolic dynamic rules of nature [46,30].

Mathematically speaking, we define a finite-state Markov Chain μ_τ with the state space $\mathcal{M}$. The chain is regarded as a parametric process for the dynamics of the system which is described by a state function x_τ and a parameter μ_τ. The parameter μ_τ may undertake a jump on the interval $(0, t)$ at times $\tau_1 < \cdots < \tau_n$, and the jumping rate is a function of time τ, state of the system x_τ, the "before-jump" value of the parameter μ_1 and the "after-jump" value of the parameter μ_2 of the Markov chain. Hence we define a function of jump rates as

$$j \stackrel{\text{def}}{=} j(\tau, x_\tau, \mu_1, \mu_2). \qquad (2.2)$$

It allows us to regard the process (x_τ, μ_τ) as a Markov process with the state space $\bar{\Sigma} = \Sigma \otimes \mathcal{M}$.

[b] The least σ-algebra that contains all open subsets of Σ.
[c] With respect to corresponding σ-algebra [19].

It should be emphasized that the system itself x_τ may not have Markovian behaviour. Thus, difficulties arise in constructing a mapping that relates the function (2.2) to states x_τ of the system. Ultimately, such difficulties stem from the problem of mathematical formalization of the concept of perturbations, which are usually regarded as a small and external-to-the-system source. Of course, in the real world modelling, statistics of the source is unknown *a priori*, which precludes assumptions based on an ϵ-additivity of perturbations. In general, such assumptions may not be adequate for the transition law of the Markov Chain as well as for the Hamiltonian of the system as a whole.

3 DYNAMIC RULES AND CONTROL PROBLEMS

Eventually, due to the approximate character of available information about the informational stream (1.6), any mathematical model can provide at best a description of a perturbed rather than an unperturbed dynamic system. Hence, if the mathematical model of a dynamic system has been constructed, in derivation of a computational algorithm we should adapt the choice of strategies s_t in our approximation of (1.6) to the character of such perturbations. Another way of putting it is that the model and the algorithm should be informationally consistent, reproducing the informational stream (1.6), and giving an approximation with a reasonable degree of accuracy.

3.1 Differential Equations and Inclusions

To include the possibility of perturbations into models let us start from the definition of a mapping

$$f(t, x_t, s_t) : T \otimes \Sigma \otimes U_T \to \mathcal{R}, \qquad (3.1)$$

where T is a given set of time. When x_t is assumed to be continuous the dynamics of a deterministic system can be appropriately described in almost-everywhere sense by the differential equation

$$x'_t = f(t, x_t, s_t), \quad x|_{t=0} = x_0^\epsilon, \ s_t \in U_T, \qquad (3.2)$$

where x_0^ϵ is an element of a given set X_ϵ defined as an ϵ-neighbourhood of an idealized point x_0. In general, the mathematical model (3.1), (3.2) can provide a description of a perturbed rather than unperturbed dynamic system. This is the case even if we formally exclude s_t from the right-hand part of the model or introduce some optimizing criteria. The next example is to demonstrate the possibility of instability in the perturbed model under any arbitrary small level of perturbations.

Example 3.1. Let us analyse unperturbed and perturbed dynamics of a homogeneous linear system:

$$(a)\ \dot{\mathbf{x}} = A\mathbf{x}, \ (b)\ \dot{\mathbf{x}}_\epsilon = A_\epsilon \mathbf{x}_\epsilon. \qquad (3.3)$$

Here we assume that the matrix A is given and $A_\epsilon = A + \Delta$, whereas $\|\Delta\| \le \epsilon\|A\|$ is the absolute error for perturbations of the matrix elements. If we assume that the initial conditions for the model (3.3) may be given precisely, then the problem of stability for the model is equivalent to the investigation of the ϵ-spectrum of the original matrix A. The ϵ-spectrum of a matrix is defined as the union of all spectra of perturbed matrices for a certain level of error [23]. In general, for any arbitrary matrix A there exists a special connection between its spectrum and its resolvent under ϵ-perturbations. The problem consists of the fact that without restrictions on ϵ, an absence of practical dichotomy can be anticipated. More precisely, there might exist such $\epsilon = \epsilon(\delta)$ that A_ϵ with $\|\Delta\| \le \epsilon$ can have in the left-half plane the number of eigenvalues different from the number of points of the matrix A spectrum. If the matrix A is defined as

$$A = (a_{ij}) = \begin{cases} -1, & j = i \ \forall i = 1, 2, \ldots, 20, \\ 10, & j = i + 1 \ \forall i = 1, 2, \ldots, 19, \\ 0, & \text{otherwise}, \end{cases}$$

and the matrix of perturbation is defined as

$$\Delta = (\delta_{ij}) = \begin{cases} \epsilon = 10^{-18}, & i = 20,\ j = 1, \\ 0, & \text{otherwise}, \end{cases}$$

then though the matrix A has one negative eigenvalue -1 of multiplicity 20, the eigenvalues of the perturbed matrix ($\sqrt[20]{10} - 1$) lie in the right-hand plane, indicating instability in the perturbed model. Of course, similar examples can be constructed for any $\epsilon > 0$ no matter how small it is assumed.

We note that Example 3.1 deals with the perturbation of the right-hand part of the model, but not with the initial condition. The latter was assumed to be fixed for both perturbed and unperturbed models. The idea of "frozen" initial conditions for a family of the perturbed right-hand parts leads to the mathematical models in which dynamic rules are defined by differential inclusions. In fact, on the basis of the point-valued map f, we can define a set-valued map [2,19]

$$\mathcal{F}(t, x_t) \stackrel{\text{def}}{=} \{f(t, x_t, s_t)\},$$

where s_t is assumed to be defined by another set-valued map. Of course, the set-valued map for the definition of s_t is coupled to the definition of the optimizing functional $F(l)$ in (1.3). Hence, when describing dynamic rules by the differential inclusion

$$x_t' \in \mathcal{F}(t, x_t) \tag{3.4}$$

in an almost-everywhere sense, a family of perturbed mathematical models (1.3), (3.4) defines an optimal control problem. In the models of this type we have a natural contradiction. On the one hand, the quality of this model has to be defined with respect to the stability of the system dynamic. On the other hand, such stability depends on the definition of s_t, which is an unknown function in the mathematical model. Hence, eventually the quality of the model depends on the definitions of the mapping (3.1) and initial conditions. In the end such definitions depend on the problem of evaluating the limit (1.12). If the initial conditions of the model are fixed, then an example of instability for the mapping (3.1) may in principle be constructed for any specified sequence s_t. This type of instability is usually referred to as computational instability. Example 3.1 clearly shows that theoretical issues of stability should primarily be addressed if "precise" initial conditions are assumed. In optimal control theory we do not require the sequence s_t to be specified explicitly, and therefore, the problem of the model stability can be formally circumvented by some appropriate regularity assumptions on the mappings $\mathcal{F}$ and F. The remaining theoretical problem is to prove that if the mapping (3.1) is well-defined then $s_0 \in U_T$, where s_0 is defined by the limit (1.12), whereas x_0 may not be given precisely. The complexity of this problem led to the constructing mathematical models of optimal control using recursive functions of density (1.10). In theory such approaches require analysis of a subsequence of (1.11) that consists of the values of the recursive function ζ,

$$(\zeta_{x_T}, \zeta_{x_{T-\Delta t}}, \ldots, \zeta_{x_t}, \ldots), \tag{3.5}$$

when $t \to 0^+$. Such analysis is typically performed for $\Delta t \to 0^+$, and essentially uses the assumptions that x_0 and ζ_{x_T} in (1.1), (3.5) may be given either precisely, or at least with equal probabilities.

First let us consider a deterministic optimal control problem where ζ_{x_t} plays the role of the value function. For the Boltza problem (1.3), (1.5), (3.1), (3.4) we can introduce the performance measure

$$J(t, x_t, s_t) = \int_t^T f_0(\tau, x_\tau, s_\tau)\, \mathrm{d}\tau + g(x(T)). \tag{3.6}$$

If we define the value function as

$$V(t, x) \stackrel{\text{def}}{=} \inf_{s_t \in U_T} J(t, x_t, s_t), \tag{3.7}$$

then using appropriate regularity assumptions and dynamic programming principle [19,20], the

original optimal control problem can be studied through the Hamilton–Jacobi–Bellman (HJB) equation

$$\dot{V}(t, x_t) + H(t, x_t, D_x V(t, x_t)) = 0, \quad V(T, \cdot) = g(\cdot), \tag{3.8}$$

where the Hamiltonian H is defined as

$$H(t, x_t, \delta) \stackrel{\text{def}}{=} \sup_{s_t \in U_T} \{-\delta \cdot f(t, x_t, s_t) - f_0(t, x_t, s_t)\}. \tag{3.9}$$

The rigour in mathematical justifications of the models (1.3), (1.5), (3.1), (3.4) and (3.6)–(3.9) is grounded in the following logical rule. Provided x_0 is given precisely, the forward-evolution model (1.3), (1.5), (3.1), (3.5) can be studied through the backward-evolution model (3.6)–(3.9) for any given function g from a specified topological space. The definition of topology for such a space requires the definition of a set in which physical states of the system can be embedded. Mathematically, the problem is usually considered with respect to Euclidean spaces (either finite dimensional [19] or infinite dimensional [28]). It allows us to use the logical rule in the reverse order: provided g is specified in a topological space, the backward-evolution model can, in principle, recover the forward evolution of the system for any given initial condition x_0.

We note that the definitions of x_0 and g are coupled to the definition of the system Hamiltonian by the specification of a topological space. An assumption that the topological space satisfies the Hausdorff separability axiom allows us to complete the chain of logical arguments in the mathematical justification of the original optimal control problem. The only problem remaining with such reasoning is that of system stability. This question is associated with the question of stability of measures defined with respect to the system's state-space, which is typically *a priori* assumed to be Hausdorff. Formally, this assumption corresponds to the choice of such a function ζ in (1.10) for which $n \to \infty$. Therefore, eventually the quality of the backward-evolution model (3.6)–(3.9) depends on the definition of a set X_ϵ from which we "puncture" a point x_0 when $\epsilon \to 0^+$. In the end, the question is reducible to the existence of an optimal strategy s_0 for such an operation, and evaluation of the limit (1.12). Since such a strategy is known neither with a deterministic certainty nor with the probability 1, it is reasonable to estimate the quality of the backward-evolution models with respect to a set X_ϵ, where ϵ may be small, but always assumed to be positive. Then the model (3.6)–(3.9) cannot be considered other than a *perturbed mathematical model*. Since $\epsilon > 0$, the instability of the system can be anticipated, unless the strategies from the set U_T are chosen consistently with the states of the system from the set Σ. Such consistency is defined by the definition of the system Hamiltonian in a chosen topological space, which is eventually defined by the mapping (3.1). In this sense the Hamiltonian can be regarded as a higher degree recursion of this mapping. Since the function $f(t, x_t, s_t)$ may be discontinuous in general, so may the Hamiltonian function, unless it can be represented as an infinite degree recursion of f. The assumption of positiveness for ϵ precludes such a situation, which seems to correspond to all physically conceivable situations. However, it implies a hyperbolicity in the underlying mathematical model [46,30]. The hyperbolic nature of mathematical models in optimal control theory stems from the splitting of the informational string (1.6) into two: (1.1) and (3.5). A simultaneous consideration of these strings implies their approximation by the perturbed informational strings

$$(x_0^\epsilon, x_1^\epsilon, \ldots, x_{t'}^\epsilon, \ldots), \tag{3.10}$$

$$(\zeta_{x_T}^\epsilon, \zeta_{x_{T-\Delta t}}^\epsilon, \ldots, \zeta_{x_t}^\epsilon). \tag{3.11}$$

After the approximation, neither of the two equalities

$$\lim_{\epsilon \to 0^+} \lim_{t' \to T} x_{t'}^\epsilon = \lim_{t' \to T} \lim_{\epsilon \to 0^+} x_{t'}^\epsilon, \tag{3.12}$$

$$\lim_{\epsilon \to 0^+} \lim_{t \to 0} \zeta_{x_t}^\epsilon = \lim_{t \to 0} \lim_{\epsilon \to 0^+} \zeta_{x_t}^\epsilon, \tag{3.13}$$

can be guaranteed in general. The lack of equalities in (3.12), (3.13) is caused by possible singularities

in transformations from s_0 to x_0^ϵ and from s_T to $\zeta_{x_T}^\epsilon$. Nevertheless, for any arbitrary $\epsilon > 0$, the informational string (1.6) can be eventually approximated as

$$(s_0, x_0^\epsilon; \zeta_{x_0}^\epsilon; s_1, x_1; \zeta_{x_1}^\epsilon; \ldots; s_{t'}, x_{t'}; \zeta_{x_{t'}}^\epsilon; \ldots), \quad (3.14)$$

when $t' \to t$, $\forall t \in (0, \infty)$. Hence, the quality of approximating (1.6) by (3.14) is defined by the sequential character of approximation for the function ζ, which in optimal control theory plays the role of the value function that depends on an approximation of the system Hamiltonian (or Lagrangian).

3.2 Stochastic Rules

Let us consider a dynamic system described in terms of the stochastic differential equation

$$\begin{aligned} \mathrm{d}x &= f(\tau, x_\tau, s_\tau)\,\mathrm{d}\tau + \sigma(\tau, x_\tau, s_\tau)\,\mathrm{d}\omega(\tau), \\ x(0) &= x_0, \end{aligned} \quad (3.15)$$

where f and σ in (3.15) denote drift and diffusion terms respectively, and ω is a Wiener process. As a functional F in (1.3) we choose

$$F(l) = E_{tx}\left\{\int_t^T f_0(\tau, x_\tau, s_\tau)\,\mathrm{d}\tau + g(x(T))\right\}. \quad (3.16)$$

Then the problem is to find

$$\inf_{U_T} F(l), \quad (3.17)$$

where $F(l)$ is defined by (3.16) under the dynamic rules (3.15), and (3.17) provides a typical example of a stochastic optimal control problem. The use of the Bellman's principle

$$\begin{aligned} V(t,x) = \inf_{U_T} E_{tx}\Bigg\{ &\int_t^{t+\Delta t} f_0(\tau, x_\tau, s_\tau)\,\mathrm{d}\tau \\ &+ V(t+\Delta t, x(t+\Delta t))\Bigg\} \end{aligned} \quad (3.18)$$

can formally reduce the problem to the dynamic programming equation

$$\begin{aligned} \min_{s_t \in U_T}[\Lambda_{s_t} V(t,x) + f_0(t, x_t, s_t)] &= 0, \\ V(T, \cdot) &= g(\cdot). \end{aligned} \quad (3.19)$$

The definition of the value function in (3.18) is analogous to that in (3.7) when we consider the conditional expectation of the performance measure (3.6). Note also that in the Eq. (3.19) the linear operator of backward evolution Λ is well-defined only if the limit

$$\Lambda V(t,x) = \lim_{\Delta t \to 0^+} \frac{E_{tx} V(t+\Delta t, x_{t+\Delta t}) - V(t, x_t)}{h} \quad (3.20)$$

exists for each $x \in \Sigma$ and $t \in I \subset [0, T]$, except of $t = T$ itself. In the end, the existence of the limit (3.20) is subject to the definition of $V(0, x_0)$. As in the deterministic case, such a definition depends on the definition of a set X_ϵ, and thus eventually requires the definition of s_0. To put it differently, for a justification of the limit in (3.20) we need existence of two limits induced by (1.10) and (3.11), namely

$$\lim_{n \to \infty} f_n(f(t, x_t)) \quad \text{and} \quad \lim_{\epsilon \to 0^+} \zeta_{x_t}^\epsilon \qquad \forall t \in [0, T].$$

The latter may be assumed *a priori* rather than justified rigorously. However, even under such an assumption the procedure of transformation from the model (3.15)–(3.17) to the model (3.19), (3.20) remains an essentially sequential heuristic procedure.

The heuristic nature of the model (3.19), (3.20) can be circumvented by using the diffusion approximation method for the original optimal control problem (3.15)–(3.17). As a result, we arrive at the form of HJB equation:

$$\dot{V} + H(t, x_t, D_x V, D_x^2 V) = 0, \quad V(T, \cdot) = g(\cdot), \quad (3.21)$$

where the Hamiltonian H is defined as

$$H(t, x_t, \delta, \Pi) \stackrel{\text{def}}{=} \sup_{s_t \in U_T} \{-\delta \cdot f(t, x_t, \delta) - \frac{1}{2}\text{tr}[\pi(t, x_t, \delta)\Pi] - f_0(t, x_t, \delta)\}. \tag{3.22}$$

Here $\pi = \sigma\sigma'$, and Π is a symmetric nonnegative definite matrix (for details, see [19]). Note that a reduction of the problem (3.15)–(3.17) to a partial differential equation by the rescaling of a Markov Chain is accompanied by a loss of information about the dynamic system itself. Indeed, the original dynamics x_t intrinsic to the model may or may not be Markovian in general. Though the Markovian property has to be preserved for the process (s_t, x_t), it may be violated after the rescaling procedure, which requires a conservation of the Markovian structure from x_t.

3.3 General Rationale for the Optimization of Singular Perturbed Dynamics

For all described dynamic rules, regularities of mappings that define the Hamiltonian of the system and the value function are coupled by a specific mathematical model, and eventually depend on the topology of the space (in which investigation of the model is being conducted) and the initial conditions of the model. In principle, *a priori* regularity assumptions on the Hamiltonian allow the recovery of information about the regularity of the sought-for solution. Results of this type provide a rigorous mathematical justification of the models for which the form of the Hamiltonian is specified. During the past years the theory has been extensively developed in this direction for deterministic and stochastic optimal control problems (see [11,47,28,19] and references therein).

Since the Hamiltonian of the system can be given only approximately, whereas regularity for the *sought-for* solution is not *a priori* knowledge being the subject of our assumptions, it seems to be reasonable to couple the model and algorithm for its solution using an approximation of the informational string (1.6). Mathematically speaking, we do not assume *a priori* "smoothness" of the "transition" between $s_{t'}$ and $x^\epsilon_{t'}$ for an approximation of the informational stream (1.6), even if $\epsilon \to 0^+$. It implies a consideration of *singular stochastic problems* in which the function x^ϵ_t is allowed to be discontinuous (the first problems of this type were studied in [3,4]). In general, since a "transition" between s_t and x^ϵ_t $(T \in (0, \infty))$ may be discontinuous, we cannot use *the principle of smooth fit* (see [54] and references therein) to claim continuity of the recursive function of density $\zeta^\epsilon_{x_t}$ when $t \to T$ (possibly $T \to \infty$). If our objective is a possibilistic attainability of the following limits

$$\lim_{\epsilon \to 0^+} x^\epsilon_t = x_t, \qquad \lim_{\epsilon \to 0^+} \zeta^\epsilon_{x_t} = \zeta_{x_t}, \tag{3.23}$$

then regularities of the limiting functions x_t and ζ_{x_t} become subject to our *a priori* assumptions, which in turn bring the possibility of singularities in such dynamic processes as "strategy-state" (s_t, x_t) and "strategy-state-density" $((s_t, x_t); \zeta_{x_t})$. It reduces the problem of analysis of the sequences (1.1) and (3.5) to the analysis of the perturbed informational strings (3.10), (3.11), which formally allows us to include the parameter of perturbation ϵ into the model. We can assume, for example, that the dynamics of the system can be effectively described by "fast" and "slow" components [59]:

$$\begin{cases} \epsilon \dot{z} = f_1(z_t, y_t, s_t, t, \epsilon), & z(0, \epsilon) = z_0, \\ \dot{y} = f_2(y_t, z_t, s_t, t, \epsilon), & y(0, \epsilon) = y_0. \end{cases} \tag{3.24}$$

If we choose a functional F in (1.3) as

$$F(l) \stackrel{\text{def}}{=} J_\epsilon = g(y_T, z_T) + \int_0^T f_0(\tau, y_\tau, z_\tau, s_\tau)\, \mathrm{d}\tau, \tag{3.25}$$

then the problem (1.3), (3.24), (3.25) is an optimal control problem for the singular perturbed dynamics. In general, neither y_t nor z_t are required to have the Markovian property. The role of the

string (s_t, x_t) in this case plays that of the sequence $(s_t, (y_t, z_t))$, in the sense that the sequence (y_t, z_t) is dependent on Markov Chain parameters, and thus the whole process $(s_t, (y_t, z_t))$ can be seen as a Markov Chain approximation. We can also interpret the sequence (y_t, z_t) when $\epsilon \to 0^+$ as the definition of a recursive function of density ζ_{x_t} with increasing degree of recurrence as $n \to \infty$. Then the model (1.3), (3.24), (3.25) will be well-defined if we define a set X_ϵ of initial conditions with a specified level of error. Hence, as above, the definition of the pair (y_0, z_0) is eventually dependent on the definition of s_0 in the informational string (1.6). It implies an approximation of the informational string (1.6) induced by singular dynamic rules using sequential decision schemes.

4 ALGORITHMIC MACHINES

Probabilistic Finite-State Finite-Action Machines Under Singular Perturbation

First, let us consider a probabilistic finite-action machine that analyses a Discrete Markov Decision process. Mathematically, the analysis can be formalized as a set of four-tuple

$$\mathcal{M} = \Big\{ x_t \in X;\ \check{s}_t \in \mathcal{U};\ \gamma_t \overset{\text{def}}{=} \gamma(x_t, \check{s}_t);\ p^\epsilon_{tt'} \overset{\text{def}}{=} p(x' = x_{t'} | (x_t, \check{s}_t)),\ x' \in X,\ t' \geq t \Big\}, \quad (4.1)$$

where $p^\epsilon_{tt'}$ is the perturbed probability of the transition from the state x_t to the next state x', γ_t is an immediate reward, $\mathcal{U}$ is a finite set of actions, X is a finite set of states, and T is a set of all times for which states from X are realizable. In general, the disturbance law of the transition probabilities in (4.1) is not known *a priori*. We may assume, however, that

$$\sum_{x' \in X} p(x' | (x_t, \check{s}_t)) = \sum_{t' \in T} p(x'_{t'} | (x_t, \check{s}_t)) = 1. \quad (4.2)$$

We also observe that every strategy s_t induces a perturbed P^ϵ_t rather than an unperturbed transition matrix. Hence, assuming the flow of time *ad-infinitum*, we can define the Cesaro-type limit matrix

$$P^\epsilon(\alpha) \overset{\text{def}}{=} \lim_{t \to \infty} \frac{1}{t} \left[P^\epsilon_0 + \sum_{k=1}^{n} P^\epsilon_{\tau_k}(\alpha) \right], \quad (4.3)$$

where $0 \leq \tau_1 \leq \tau_2 \cdots \leq \tau_n < t$ with the possibility of $n \to \infty$. A strategy α in (4.3) denotes a sequence that consists of elements s_t. Of course, using the reward function $\gamma(\cdot, \cdot)$, we can construct classes of optimization problems in a way similar to what we have done with respect to the loss function in Section 1. For example, we can consider the limit Markov control problem

$$J^\epsilon(\check{s}, s_t) \to \max, \quad s_t \in U_T, \quad (4.4)$$

where

$$J^\epsilon(\check{s}, s_t) \overset{\text{def}}{=} \liminf_{t \to \infty} \frac{1}{t} \left[E_\alpha(\gamma_0, \check{s}) + \sum_{k=1}^{n} E_\alpha(\gamma_t, \check{s}) \right], \quad (4.5)$$

and $\check{s} \in \mathcal{U}$, $\alpha \in U_T$. We note that the definition of the matrix P^ϵ_0 in (4.3) and the quantity $E_\alpha(\gamma_0, \check{s})$ in the problem (4.4), (4.5) eventually depends on our definition of the first pair (s_0, x_0) in the informational stream (1.6), which may be given only approximately. Hence, it is reasonable to assume that the transition law matrix P^ϵ_0 has Markovian structure under specified n if the exact equality in (4.2) holds. To put it differently, for any finite n the structure of P^ϵ_0 depends on the topological structure of sets X and T, thus when X and T are specified such dependency remains in force even if $n \to \infty$. In the general case, it precludes the definition of the matrix P^ϵ_0 as a fixed finite dimensional matrix with the probability 1 [16]. As a result, stability analysis of the associated optimization models requires consideration of a family of matrices P^ϵ_0 under a specified level of error. Recall that a similar situation holds when dynamic rules are given. Then, we need the whole set X_ϵ under a specified level of error to perform analysis of stability. Without such a "relaxation" of

probabilistic requirements on the initial conditions of the model, for any arbitrary small $\epsilon > 0$ an example of practical instability can always be constructed.

Deterministic Finite-State Finite-Memory Machines

Now let us consider another type of algorithmic machine. Deterministic finite-state machines in the case of finite memory are defined as the triple [42]

$$\mathcal{D} = (\Sigma_m, f_2, f_1), \tag{4.6}$$

where Σ_m is a finite set of machine states, and f_1 is a mapping $\Sigma_s \otimes \Sigma_m \to \Sigma_m$ which defines the machine-next-state function. The set Σ_s is a finite set of system states. More precisely, we assume that Σ_s can be formalized as a sequence (1.1) as a result of observations, computations, measurements etc. This sequence "feeds" the machine (4.6). The mapping $f_2 : \Sigma_m \to U_T$ defines the output function with a set of strategies U_T. Hence, starting from the state $\hat{s}_0 \in \Sigma_m$, the machine (4.6) produces strategies $(s_1, s_2, \ldots)$ while going through a sequence of its states $(\hat{s}_1, \hat{s}_2, \ldots)$ according to the recursive rules

$$\hat{s}_t = f_1(x_{t-1}, \hat{s}_{t-1}), \qquad s_t = f_2(\hat{s}_t). \tag{4.7}$$

Excluding the current state of the machine $\hat{s}_t$ from (4.7), we find a function of strategies as a second degree recursion of the sequence $(x_{t-1}, \hat{s}_{t-1})$

$$s_t = f_2(f_1(x_{t-1}, \hat{s}_{t-1})). \tag{4.8}$$

Hence, having knowledge of the previous state of the machine and a corresponding letter of the alphabet Σ_s, we can define the current strategy using the recursive function (4.8). This model does not require any formal association with a statistical model, and does not even assume the existence of the latter [42]. The informational data stream produced by such machine is

$$((x_0, \hat{s}_0), s_1, (x_1, \hat{s}_1), \ldots). \tag{4.9}$$

From (4.9) we conclude that the starting information to compute the first strategy is a pair $(x_0, \hat{s}_0)$. We also observe that the main drawback of such a deterministic model is the requirement to fix the strategy immediately when the state of the machine $\mathcal{D}$ is given. Loosely speaking, some relaxation time between the transition $\hat{s}_{t-1} \to \hat{s}_t$ should be incorporated into the model to allow strategy correction. Indeed, such time is implemented into probabilistic finite-state finite-action machines by probabilities of the transition from one state of the system to another under certain actions of a controller or DM. However, if we know *a priori* that

$$P(\hat{s}_{t-1} \to \hat{s}_t | x_{t-1}, (s_t, x_t)) = 1, \tag{4.10}$$

or time for such a transition is defined by a given time-interval, then the sequential decision scheme based on deterministic finite-state finite-memory machines is quite natural. If such information is not available *a priori*, then probabilistic finite-state finite-action machines appear to be useful in the analysis of system dynamics.

In the next sections we develop a technique to find a reasonable compromise between the two approaches described above.

5 THE PERTURBATION PARAMETER AS A FUZZY BORDER BETWEEN DETERMINISTIC AND PROBABILISTIC DESCRIPTIONS OF SYSTEM DYNAMICS

Major complexity in the mathematical modelling of dynamic systems arise from the *a priori* unknown character of the disturbance law. On one hand, the implicit assumption of deterministic models on the existence of an associated optimal algorithm (like an assumption (4.10)) can be hardly justified in modelling complex processes and phenomena. On the other hand, the main difficulty in effective applications of probabilistic models arises from the question of how common is the ergodicity of the Hamiltonian flow on the energy surface [24]. As was pointed out, perturbations can *qualitatively* change the ergodic structure of the

underlying dynamic system. The examples of Markov Chains with discontinuities in the stationary distribution of the perturbed system can be found, for example, in [52,1]. Furthermore, for any decomposition of such a chain into a finite number of independent ergodic subclasses (under the assumption $\epsilon \to 0^+$) examples of system instability can be constructed for arbitrary small ϵ.

5.1 Degree of Recurrence in Mathematical Models for Evolution

An idealization of "unperturbed" mathematical models obtained in the limit of vanishing perturbations $\epsilon \to 0^+$ can often help to better understand real-world phenomena and processes. However, it should be realized that such an idealization has limited applicability, and depends on quite restrictive mathematical assumptions related to

- homogeneity of the environment of the system, and
- uniformity of density which characterizes the system or its parts.

Since for any model of a dynamic system with specified dynamic rules the parameter of perturbation ϵ may be small but always positive, rescaling procedures for the associated (with the optimization model) Markov Chain may not provide an adequate approximation to the system dynamic. Such procedures may eventually ignore the neighborhood structure of the chain. If such a rescaling (for example the diffusion approximation) has been performed, then the original problem can be reformulated as an inverse problem with respect to a recursive function of density (1.10). The complexity of the solution of the inverse problem is determined by the degree of recurrence n and the topology of the space where investigation is being conducted. Moreover, if the topology is *a priori* specified then the regularity assumptions on the function f_n allow us to recover the information on the regularity of the function ζ, at least in principle for any arbitrarily big n, following certain logical rules. In the models like (3.8), (3.9) and (3.21), (3.22), f_n plays the role of the Hamiltonian function. Such models can be regarded as discrete optimization problems if we interpret the function f_n as one that defines the top-level goal, whereas all functions f_i, $i = n-1, \ldots, 1$ are supposed to define certain subgoals. The definition of the density function provides constraints for such a problem of multicriteria optimization. From the physical point of view such problems require finding the minimum of the Hamiltonian of the system on the energy surface, and can be formulated as follows: given a finite (typically large) number n of subsystems of a big system, minimize an *approximation* to the system Hamiltonian on an approximating set of its energy surface.

Now recall the definition of system entropy in statistical physics as a quantity that is uncertain to an additive constant and is dependent on the choice of units, defined by the Liouville measure [36]

$$\sigma = -\int f \log[(2\pi\hbar)^s f]\, \mathrm{d}p\, \mathrm{d}q. \tag{5.1}$$

Here s is the degree of system freedom, p and q are momentum and position variables. If we assume that the whole system entropy can be defined through the entropies of its subsystems as $\sigma = \Sigma_i \sigma_i$, then for any probability distribution $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ its associated information can be defined as the Shannon entropy [53,49]:

$$\sigma_s(\mathbf{p}) = -\sum_{i=1}^{n} p_i \log p_i. \tag{5.2}$$

The constant n in (5.2) can be approximated with respect to the required accuracy ϵ and is ultimately coupled to the definition of s in (5.1). In the limit of "vanishing perturbations" $\epsilon \to 0^+$ and "maximum knowledge" $n \to \infty$, the Shannon entropy can be generalized to the continuous case of the Boltzmann–Gibbs entropy. The latter transformation requires a justification of system stability. From the physical perspective mathematical idealization of two simultaneous limits $n \to \infty$ and $\epsilon \to 0^+$ requires an estimation of the degree of system freedom in the definition (5.1). In this sense

such an idealization is problem specific, and always requires analysis of the measure stability.

5.2 Discrete Optimization and Evolution of Thermodynamic Systems

Any specific algorithm for the solution of the problem of modelling dynamic system evolution is affected by the form of the function f_n (as a Hamiltonian approximation on the energy surface) and by the neighbourhood structure of the system evolution. In this sense an algorithm is always coupled to the problem specific information. In discrete optimization such algorithms can be conditionally divided into three main categories [10,50]:

- *constructive algorithms* (*CAs*) that require construction of decreasing and embedded in each other subsets of a given finite set of states Σ,
- *sequential algorithms* (*SAs*) that attempt to construct a path through Σ, and
- *evolutionary algorithms* (*EAs*) that manipulate sets of solutions in Σ.

Let us assume that, for any given state x_t from Σ that characterizes the whole system, there is a neighbouring set of states N_{x_t} where transitions from x_t are allowed. Then CAs usually apply a "greedy" policy – when starting from $x_0 \in \Sigma$, they choose at stage n an x_{n+1} such that

$$\mathcal{E}(x_{n+1}) = \min\{\mathcal{E}(t):\ t \in N_{x_n}\}, \qquad (5.3)$$

where $\mathcal{E}$ is an energy functional. Mathematically speaking, we expect that given $\mathcal{E}$ and an accuracy $\epsilon > 0$, we can find a solution, at least in principle, when $n \to \infty$. However, it is well-known that as a result of such policy CAs may relatively easily be trapped in a local minimum of $\mathcal{E}$. If $\mathcal{E}$ is assumed to be continuous and Σ is a "reach" enough set, then in general the degree of recursion in (1.10) tends to infinity and we theoretically face infinitely many optimization problems (5.3). By now it is clear that without an appropriate analysis of the structure N_{x_n}, success of such algorithms cannot be guaranteed. As we pointed out earlier, such analysis has to be conducted with respect to given ϵ.

The main advantage of SAs is based on the fact that they do not exclude the theoretical possibility of occasional acceptance of new states that may increase the energy functional [43]. We also assume that an "initial" solution $x_0 \in \Sigma$ may be given (for example, obtained by a CA). Moving to a neighbouring solution $x' \in \Sigma$, the structure of the neighbourhood of the solution should be carefully analysed to avoid the difficulty of CAs[d]. The basic idea for such an analysis came from statistical physics. The growing complexity of the solution of deterministic equations of motion for a system of many subsystems (such as particles) has led to the idea of ensemble averaging instead of classic-mechanical averaging in time. As the number of subsystems increases dramatically, the Monte-Carlo and particle-type simulations [27] eventually remain the only algorithmic procedures that can be applied in theoretical generality. However, such procedures may encounter serious difficulties in non-equilibrium thermodynamics [48]. In a search for alternative approaches to the ensemble averaging, many useful ideas have been generated during recent years. The intrinsic ability of Markov Chains to form a canonical Gibbs ensemble numerically has led to growing interest in the subject [19,35]. Using the principles of statistical physics we can assign to each state $x_t \in \Sigma$ the probability

$$p_T(x_t) = \frac{\exp(-f(x_t)/T)}{\sum_{x_t \in \mathcal{A}} \exp(-f(x_t)/T)}, \qquad (5.4)$$

where $f(x_t) = \mathcal{E}(x_t)/k$. The quantity $\mathcal{E}(x_t)$ can be interpreted as the potential energy of each state (or subsystem) in phase space that belongs to an ensemble. The probability that a system belongs to the ensemble is proportional to $\exp[-\mathcal{E}/(kT)]$ where k is the Boltzmann constant. We observe that the smaller $T > 0$ is, the more evident is the tendency of the Gibbs distribution defined by (5.4) to be concentrated on states x_t with small values of $f(x_t)$. Hence, if we could simulate the *cooling* of the

[d] There are classic examples of SAs like the steepest-descent method that have potentially the same problems as CAs.

system, a state of minimum energy may, in principle, be obtained provided that the Markov Chain converges (in distribution) to the Gibbs distribution (stationary) law. This allows us to consider CAs as a partial case of this general interpretation when a Markov Chain is run for $T \to 0^+$. Another extreme case of the "high T limit" ultimately leads to the idea of *dynamic continuity*. In such a case all states are assigned the same probability, and evolution is thought as moving from a state to its neighbours *uniformly*. The computational implementation of the above idea is provided by the simulated annealing algorithm first proposed in [31]. For a real physical system, temperature may be lowered too rapidly, and the system may be trapped in a local energy minimum. However, the choice of $T_n = c/\log n$ with a sufficiently large c can theoretically guarantee the system's "escape" from the local minimum [21]. In practice, the algorithm works as follows. If for the time-index n x_{t_n} is given, then from the set $N_{x_{t_n}}$ we choose state t, calculate $\Delta f = f(t) - f(x_{t_n})$, and set

$$x_{t_{n+1}} = \begin{cases} t, & \text{with probability } p = \exp(-\bar{\Delta}/T_n), \\ x_{t_n}, & \text{with probability } 1 - p, \end{cases}$$

where $\bar{\Delta}$ is Δ when Δ is positive and zero otherwise. Of course, the choice of the *neighbourhood structure* is crucially important for the algorithm's performance. If the neighbourhood is chosen too small, then the resulting simulated Markov Chain may move very slowly around Σ in the search of the minimum. On the other hand, if the neighbourhood is chosen too large, then the process eventually performs a "blind" random search throughout Σ. It samples randomly from a large portion of the state space, and every next possible state is chosen practically uniformly over the whole set Σ. As an extreme case it may happen that $N_{x_t} = \Sigma$. The conclusion which has to be drawn from the above consideration is that the choice of neighbourhood should be adapted to the approximation of the energy functional (or system Hamiltonian) in the search for a compromise between these two extremes.

The first step towards such an adaptation is realized in EAs. Typically, EAs deal with a population of solution instead of a single partial solution, as in CAs or SAs. The most important advantage of EAs consists of allowing an exchange of information between solutions in the current population (a cooperation step during the "generation cycle"). The main problems for EAs are related to the self-adaptation step when the solution's internal structure may be changed without interaction with other members of the population. When there are a lot of replicates of the same solution in a population, EAs may converge prematurely, which is usually called a diversity crisis. In such situations EAs are not competitive with the best versions of SAs.

Let us summarise the definitions of strategies in the above three classes of discrete optimization algorithms:

$$\begin{cases} s_t = \mathcal{F}_1(x_{t-\Delta t}, \mathcal{E}) & \text{for CA}, \\ s_t = \mathcal{F}_2(x_{t-\Delta t}, N_{x_t-\Delta t}, \mathcal{E}) & \text{for SA}, \\ (X_n^\epsilon, s_t),\ s_t = \mathcal{F}_3(N_{x_t-\Delta t}, \mathcal{E}) & \text{for EA}. \end{cases} \tag{5.5}$$

Here $\Delta t > 0$ is a relaxation time coupled to the algorithm performance when $\epsilon > 0$, and X_n^ϵ is a population of solutions for the nth generating cycle. Functions F_i, $i = 1, 2, 3$ are algorithm-specific. In general, they can be regarded as recursive functions of energy functionals, and the set of initial approximations X_ϵ for the specific algorithm:

$$F_i = f_{n_i}(f_{n_i-1}(\ldots(f_1(X_\epsilon, \varepsilon)\ldots)). \tag{5.6}$$

At any specified moment of time t, the definition of strategy s_t implies a coupling rule between ϵ and n_i. The definition of such a coupling leads to the well-posedness of the problem. In this sense, the well-posedness of limiting models based on the assumptions $\epsilon \to \infty$ and $n_i \to \infty$ is totally dependent on *complete information* about the initial conditions of the system, and a precise definition of the energy functional.

The process of constructing mathematical models is always a *competition* between (i) *an approximation of the system–environment boundary*

interface (which involves the system's internal time [44]), and (ii) the conservation laws for integral characteristics of the system (which involves modeler's time [39]). As a result of such a competition, the resulting mathematical models *simulate* coupling of the system to its environment, and can be considered as models of *neither isolated nor closed systems*. A formal expression of the competition is provided by the physical concept of relaxation time. Having captured in the mathematical model the notion of information formally, its numerical expressions can be used in decision making with uncertainty, characterised by the adequacy of the simulation of the system–environment coupling. In general, numerico-logical methods can be used effectively only if an appropriate model has been constructed. Hence, the quality of an algorithm depends decisively on an adequate reflection of the system–environment coupling in the mathematical model. If constructing a model is an art rather than a science, then the latter formally begins from the derivation of an algorithm from the model [56].

In concluding this section, it should be emphasized that the quality of a mathematical model for dynamic system evolution is decisively dependent on (i) the approximation of the initial conditions for the system, and (ii) the approximation of the system–environment boundary interface. To minimize such dependency, the solution of a sequence of optimization problems can be used as an alternative to the limiting rescaling procedures approach. Such an approach seems to be more physically reasonable, since *a priori* information about the system can be given only as a certain possibilistic distribution which allows us to select a new distribution according to certain principles [15,49].

6 COUPLED MATHEMATICAL MODELS OF MACRO- AND MICRO-EVOLUTION

The complexity in identifying a "hard boundary" interaction between system and its environment is eventually determined by the degree of recurrence in the definition of the system Hamiltonian. Such a definition should be given with respect to the upper bound of error ϵ in the identification of the set of initial conditions X_ϵ. Since, in general, perturbed and unperturbed models might give rise to *qualitatively distinct types of descriptions of system behaviour* for any arbitrary $\epsilon > 0$, the perturbation parameter alone cannot be an appropriate characteristic of the model's uncertainty. We observe that perturbations are an important part of the system dynamics which cannot be appropriately formalized in mathematical models unless we regard the mathematical modelling of dynamic system evolution as a *decision making process with limited information* from the very beginning of the modelling process. Additional information about the system becomes available in time at stages due to the model-associated computations, observations and measurements. Hence, to approximate the dynamic system evolution, it is essential to take into consideration the fact that initial information about the system can only be given approximately. A mathematical formalization of such approximations is a challenging problem that requires new approaches.

On one hand, the idea of sequential approximation and the hyperbolicity of the underlying differential equations is an intrinsic element of recent investigations in physics foundations [46,30]. On the other hand, rescaling procedures allow us to construct mathematical models which are essentially parabolic by their nature. Moreover, the latter have proved to be a very useful tool for investigating the laws of nature. Although such rescaling procedures are always connected with the loss of some information, a justification of parabolic approximations of dynamic system evolution may be obtained if we assume that there exists a system density f on the Gibbs phase space Γ such that its associated index of probability is given by $\log f$. In general it allows us to consider the definition of entropy in the Gibbs form as

$$H(f) = -\int_\Gamma \eta(f)\mu_\varepsilon(\mathrm{d}x_t) \qquad (6.1)$$

instead of the definition (5.1), where η is defined by (1.9). Such a formal identification of a (thermo)-dynamic system with a probability space is based on the Gibbs conjecture. Namely, we assume that the appropriate description of a macroscopic system in thermodynamic equilibrium may be provided by certain probability measures on the phase space of the system. Although this conjecture has never been rigorously proved [24,39,40], the passage from (5.1) to (6.1) is not without certain gains. It provides a convenient framework for the development of a mathematical theory for dynamic systems allowing the formulation of the concept of ergodic theory that expresses at least some aspects of irreversible thermodynamic evolution [45]. However, the introduction of a recursion function ζ using the Lebesgue measure $\mu_{\mathcal{E}}(\mathrm{d}x_t)$ does not answer the question of stability for a "projection" of the Liouville measure (for a system with a certain degree of freedom (5.1)) onto the energy surface using a sequence of the Gibbs measures that deal with microcanonical ensembles. As we explained above, from the physical point of view we should approximate the system Hamiltonian on the energy surface, which is also subject to an approximation. Hence, mathematically speaking, to rigorously justify models arising from application of the Gibbs conjecture, we should be able to construct both the forward-evolution model and its associate for the backward-evolution as we explained it in Section 3. Gibbs was the first who arrived at the concept of mixing, and who noticed that the very use of probabilities in the description of physical states implies a time asymmetry [45]. In turn, the latter implies reversibility of distribution functions in a mathematical sense, as well as a forgetfulness property with respect to the initial conditions of the system in the flow of time. Such a reversible time-asymmetry in the mathematical theory of dynamic systems is in contrast with the irreversible character of evolution implied by the second law of thermodynamics and Eddington's time arrow. The complexity of the mathematical formalization of evolution irreversibility was well understood by J. Gibbs, who wrote [22],

> it should not be forgotten when ensembles are chosen to illustrate the probabilities of events in the real world, that while the probabilities of subsequent events may often be determined from the probabilities of prior events, it is rarely the case that probabilities of prior events can be determined from those of subsequent events, for we are rarely justified in *excluding* the considerations of the antecedent probability of the prior events.

Almost a century ago he clearly pinpointed that the main difficulty in a mathematical formalization of the backward evolution models lies in the complexity of a probabilistic description of the initial conditions for the dynamic system, even if the probability of a terminal event is assumed to be given *a priori*. At the same time he proposed an approach that allows the effective construction of a framework for a formal separation of the "observer" from the "modeller", and the system from its environment. Such a construction plays a resolving role in mathematical modelling and computational experiments. In fact, if the conjecture is accepted, the "modeller" (at least in principle) can perform a task in the "best" possible way, and the idea to exclude the "observer" from *the intermediate process of computations* (except at the very beginning and the very end of this process) becomes natural [60]. Then the whole time-set of the evolution of a dynamic system may be associated exclusively with the "modeller" as an "error-nulling" optimizing device. The existence of such a device depends on the existence of an error-free model of dynamic systems, that in turn eventually depends on the definition of a sequence of switching events or a time-partition, when the "modeller" may become the "observer" and vice versa.

Starting from this idea we can introduce the notion of a Generalized Dynamic System (GDS) where the decision maker (modeller/observer or problem solver) is considered as an intrinsic part of the model [39]. The basic steps of such a model construction are as follows: first, we consider the

mathematical model of a dynamic system

$$e_{n+1} = H(v_\epsilon, e_n), \quad n = 0, 1, \ldots \tag{6.2}$$

as a mapping that couples two space–time events of the system evolution by a function of the perturbed velocity v_ϵ and the system's Hamiltonian or its approximation H. Then, we specify a sequence of events $(e_0, e_1, \ldots)$ by temporal evolution. In practice such a specification is always an approximation for both the probabilistic and deterministic approaches. We assume that the basic features of dynamic rules that govern a system can be appropriately described by a velocity function v_1. Furthermore, we allow the possibility of a "correction" of these dynamic rules by another dynamic which is specified by another velocity function v_0. Formally, v_1 can be seen as a higher, but *a priori* unknown, degree of recursion of the function v_0. As a result, we arrive at the two coupled sequences

$$(x_0, x_1, \ldots) \quad \text{and} \quad (h_0, h_1, \ldots). \tag{6.3}$$

When $n \to \infty$ and $\epsilon \to 0^+$ we expect that the sequences (6.3) merge, producing events that can be characterized by the limit of the model (6.2). Since neither the degree of recursion nor the level of perturbations are known *a priori*, we formalize the dynamics of the system by the two equations

$$\begin{cases} x_{t+1} = H_1(v_1, x_t), \\ h_{\tau+1} = H_0(v_0, h_\tau), \end{cases} \tag{6.4}$$

where H_1 is an approximation to H and H_0 is an operator for sequential corrections of such an approximation. If we assume that in principle system dynamics can be described with arbitrary accuracy, then the first equation of the system (6.4) in the long run should be practically independent of v_0. Such a limiting case corresponds to viewing perturbations as a force, "continuously" external to the system. However, in general, both functions v_0 and v_1 are perturbation-dependent. Thus, the system (6.4) provides the possibility of looking at the coupling between the velocity of the perturbed system and perturbations of its environment. It is assumed that in general such coupling can be looked at in two different space–time frames of reference, macroscopic and microscopic.

One possible direction in the development of the theory of dynamic systems was provided by the celebrated Gibbs conjecture which we mentioned above. This led naturally to the idea of the control of dynamics described adequately (for example, in the almost-everywhere sense) by the first equation of the system (6.4) or its consequences, some of which we have considered in previous sections. Under this approach mathematical formalization of the decision rules need some *a priori* assumptions on the smoothness of the function (or functions) that provides (or provide) an approximation to the recursive function H_1. It is precisely these assumptions which formally allow the use of the perturbation theory in the investigation of underlying dynamic problems. In this way we "localize" the problem of scale interactions into a perturbation parameter ϵ which stores information about the complexity of the problem no matter how big the degree of recursion n really is. From this point of view it seems reasonable to look at the classical system of the theory of singular perturbations (like (1.3), (3.24), (3.25)) as those that may be obtained as a partial case of (6.4) by some appropriate rescaling procedures. More precisely, if ϵ is interpreted as a force, which is external to the system, then in the limit of $\epsilon \to 0^+$ the classic models in the theory of singular perturbations may be regarded as an infinite-recursion decision rule.

In the general case, however, the model (6.4) provides an interpretation of perturbations as an intrinsic to-the-system force. In this case it is reasonable to assume that both functions v_0 and v_1 are dependent on ϵ for any interval of time. Moreover, since the only available *a priori* information on ϵ is its positiveness, we need to introduce a mapping to describe the behaviour of ϵ while the system evolves. To put it differently, in order to perform at least in principle an infinite-recursion procedure when $\epsilon \to 0^+$ and $n \to \infty$, we need some

learning rules to be introduced into the model. In [39] it was shown that under quite general assumptions the optimal control problem (1.3), (1.5), (3.2) is reducible to the hyperbolic-type equation (generalized energy equation):

$$(1+v_1)\left[\frac{\partial\mu}{\partial x}+\frac{1}{v_1}\left(\frac{\partial\mu}{\partial t}+f_0\right)\right]=0, \tag{6.5}$$

that has a unique generalized solution (in the sense of an integral identity). The unknown function was assumed to be Lebesgue integrable, that is $\mu\in L^1(Q)$, where Q is the space–time region of interest. In the general case this function is referred to as the *decision maker* function. The interpretation of Eq. (6.5) as a partial case of the system (6.4) can be formally given as follows. We consider a mathematical model that consists of two parts: (i) an idealized equation for a phase point in the system's time (with a trajectory $h(\tau)$) associated with the centre of the system gravity, and (ii) the macro-model of dynamic system micro-evolution in the decision-maker time "external" to the system (in terms of the decision-maker function μ)[e]. Such a model of a Generalized Dynamic System couples two different space–time scales with the perturbed velocity function v_ϵ in its two different manifestations, micro-velocity v_0, and macro-velocity v_1:

$$\begin{cases}\dot{h}(\tau)=v_0(\tau,h,\mu), & \text{(6.6a)}\\ \partial\mu/\partial t+v_1(t,x,\mu)\partial\mu/\partial x=0. & \text{(6.6b)}\end{cases}$$

Hence, the model is constructed in such a way that both parts of the perturbed velocity functions v_0 and v_1 inherit their dependency on the decision-maker function. If two events (between which GDS evolution has to be studied) are specified, then *a pair of functions* $(h(\tau),\mu(t,x))$ gives the solution to the problem. An approximation of such events can be given using a probabilistic connection between the micro and macro levels of the system description in the form of the complementarity principle

$$v_0(h,\tau,\mu)|_{\tau=\tau_0}\cdot\mu(t,x,v_1)|_{t=t_0}=1. \tag{6.7}$$

If the smaller velocity v_0 is assumed, then the bigger μ at the initial moment of time should be chosen. Hence, formally by (6.7), we postulate the existence of the system in a space–time of events with the probability 1 at the initial moment t_0 of absolute DM-time for any arbitrary small values of v_0. Since τ_0 may be given only approximately, any approximation that follows from (6.6), (6.7) enables us to identify such an approximation with a Perturbed GDS (PGDS). In the limit of vanishing perturbations ($\epsilon\to 0^+$) the model (6.6), (6.7) (PGDS evolution) formally converts into the model for Unperturbed GDS (UGDS) evolution and merges with the model (6.2). Therefore, in principle the model (6.5) can be obtained from (6.6), (6.7) using (6.6(b)) as a corrector for Eq. (6.6(a)). Such a corrector induces the presence in Eq. (6.5) of the goal function f_0. The main difficulty behind such a formal procedure is how to construct an appropriate corrector. From the probabilistic point of view this difficulty was dealt with by Gibbs. Of course, there do not exist two non-identical events (related to the present state of the system evolution, and its future or past behaviour) described by any mathematical model with the same probability exactly equal to 1. In reality, all constructions of mathematical models for dynamic system evolution start from a countable base in space–time of events of PGDS evolution. At the next step, we approximate (6.2), and this "fuzzifies" the deterministic concepts of evolution in the probabilistic descriptions of events. It should be noted, however, that a *randomness of GDS evolution* is induced by inherent approximations in the model construction and is not an independently established fact by itself. The lack of rigour in the description of a dynamic system by purely probabilistic models stems from the fact of such an approximation. On

[e] We started from the consideration of the equations $\dot{h}=v_0(\tau,h,\mu)$ and $\dot{x}=-v_1(t,x,\mu)$.

the other hand, the main difficulty in applications of deterministic models is in the construction of effective correctors to describe adequately dynamic rules. In both situations the success of modelling is defined by the quality of an algorithm, which should be derived from the model using the concept of system stability.

7 COMPUTATIONAL MODELS AS MARKOV CHAIN APPROXIMATIONS

As soon as dynamic rules (with or without control) define a model for system evolution as a function of time x_t^ϵ, such a function becomes subject to intrinsic uncertainty for arbitrary small intervals of time. This is a natural reflection of the approximate character of mathematical models which can be in principle characterized by the degree n of recursion for such a function with respect to the function of density. Since such a degree can be rarely given *a priori*, we can approach the problem solution by imposing an upper bound on ϵ. It seems to be natural that in applications to the real world, mathematical models of dynamic systems have to be understood as perturbed rather than unperturbed models. Of course, they will remain as such in the foreseeable future. In general, it precludes assumptions on the forgetfulness property for density distributions, and as a result the Markovian property for the perturbed system dynamics x_t^ϵ. Behind the complexity of the problem is the question of the system's stability. The idea which will be developed in what follows is to construct a Markov Chain approximation *simultaneously* with an approximation of the system (that depends on Markov Chain parameters) to guarantee its stability. Hence *the Markov Chain shall play the role of a learning rule for the system* under an approximation of the perturbed system's velocity by its approximation v_1 in the macroscopic DM frame of reference. As a result of such a construction and the Markov theorem on the generalized law of big numbers, the pair of functions $(h(\tau), \mu(t, x))$, which describes the process of GDS evolution, shall possess the Markovian property. Furthermore, it is proposed to approximate this process by a pair of discrete functions $(\xi_n^{\tau h}, \mu_n^{\tau h})$[f], where $\xi_n^{\tau h}$ is an associated (with the microscopic frame of reference) Markov Chain state.

Let us consider the PGDS described by the form of the generalized energy equation (6.5)

$$\frac{\partial \mu}{\partial t} + v_1(t, x, \mu)\frac{\partial \mu}{\partial x} = \tilde{f}_0(t, x, \mu). \tag{7.1}$$

The approximation of the initial condition for this model is specified in the DM-time scale as

$$\mu(x, t)|_{t=t_0} = \delta(\epsilon), \tag{7.2}$$

where ϵ depends on the approximation of the function v_0 in (6.7). Hence, formally, the model (7.1), (7.2) can be seen as a macro-model for GDS evolution. However, microscopic features of the dynamics[g] are taken into account by the possibility of coupling between the parameter of system perturbations ϵ and the decision-maker function μ. In what follows, a technique which is based on the construction of a hybrid-type algorithm [10] for the solution of this problem will be developed. The main results concern the derivation of a learning heuristic procedure that combine the effective features of (5.5), (5.6). To simplify the derivation, I explain the main ideas in the one-dimensional case, denoting a characteristic length of the system as h and assuming that $h \ll T - t_0$. Let us consider the evolution of the system defined by the dynamic rules (7.1), (7.2) in a square region of the macroscopic frame of reference

$$\bar{G} = \{(x, t): \quad x_0^\epsilon \le x < X_t, \quad t_0^\epsilon \le t < T_x\}, \tag{7.3}$$

where absolute DM-times of initial $(t_0^\epsilon = t_0)$ and terminal $(T_x = T)$ events, as well as a position $x_0^\epsilon = x_0$ of the system, are specified. If GDS

[f] Compared to random processes with Markov Chain parameters in the continuous absolute time in [13,19] and references therein.
[g] Induced by (i) an approximation of system–environment boundaries at $t = t_0$ and (ii) corrections of the function v_1 by v_0.

evolution takes place in $\bar{G}$ under a certain level of perturbations $\epsilon > 0$, then for this region the function v_1 depends on the DM-function μ. This depends on v_0 being subject to approximation from the initial moment of DM-time. Hence, we shall approximate the function v_1 with respect to our approximation of the function v_0 in a recursive manner. First we introduce the discrete grid in the region (7.3):

$$\omega_{\tau h} = \Big\{(x_i, t^j):\ x_{i+1} = x_i + h_i,\ t^{j+1} = t^j + \tau_j,$$
$$i = \overline{0, n-1},\ j = \overline{0, m-1},\ t^m = T\Big\}, \qquad (7.4)$$

and consider an elementary space–time cell $c_{ij} = [x_i, x_{i+1}] \otimes [t^j, t^{j+1}] \subset \bar{G}$. The nodes of the grid (7.4) connect events relevant to the system evolution. We shall refer to the whole set of such events in $\bar{G}$ as a set of macroscopic events. Let t^j and t^{j+1} be two moments of absolute time (defined by DM) that correspond to two subsequent macroscopic events e_j, e_{j+1} of system evolution. Since the process (x_t, μ_t) is assumed to be Markovian, these events can be specified by two pairs of discrete functions $e_j = (\xi_j^{\tau h}, \mu(x_i, t^j))$, $e_{j+1} = (\xi_{j+1}^{\tau h}, \mu(x_{i+1}, t^{j+1}))$, where $\xi_j^{\tau h} = x_i^j$ and $\xi_{j+1}^{\tau h} = x_{i+1}^{j+1}$ are states of the associated Markov Chain[h]. To preserve basic macroscopic features of the system, the values of jumps $\Delta\xi_j^{\tau h} = \xi_{j+1}^{\tau h} - \xi_j^{\tau h}$ of this chain should be subordinated to the corresponding approximation of system–environment boundaries. For example, let the time spent to cover the characteristic length h of the system be τ. Then, we formally express the idea of subordination in the definition which follows, where we consider the limiting case $\tau \to 0$ of such a subordination.

DEFINITION 7.1 Let $e_j = (\xi_j^{\tau h}, \mu(x_i, t^j))$, $e_{j+1} = (\xi_{j+1}^{\tau h}, \mu(x_{i+1}, t^{j+1}))$ be two subsequent macroscopic events of GDS evolution that happen with the probability 1. Then the *GDS velocity function between the macroscopic events* e_j and e_{j+1} can be defined in an elementary space–time cell $c_{ij} \subset \bar{G}$ as

$$v(t, x) = \lim_{\tau \to 0} \frac{E^{\tau h | (x_i, \mu^j)} \Delta\xi_j^{\tau h}}{\tau}. \qquad (7.5)$$

The numerator under the limit in (7.5) is referred to as *the velocity of the Markov Chain between two subsequent macroscopic events.*

The definition of the velocity function as the most probable jump of the associated Markov Chain (the jump which minimizes the energy of the transition) gives a way to construct a stable approximation of the Hamiltonian of GDS evolution. We relate the macroscopic behaviour of the system to its microscopic characteristics defined in an elementary space–time cell c_{ij}. As a result, in any such cell[i] the GDS velocity defined by (7.5) is always greater than or equal to 1. Hence, if the process is approximated in c_{ij}, the Courant–Friedrichs–Lewy (CFL) stability condition [12] ($\tau \le h$) is satisfied automatically, regardless of the actual values of the velocity function in c_{ij}.

Remark 7.1 In the limiting case $h \to 0$, Definition 7.1 loses its meaning and a macroscopic system degenerates into a point. Mathematically, however, this situation is well-defined as $n \to \infty$ ($m \ge n$):

$$\lim_{n \to \infty} v(t, x) = v_\epsilon, \qquad (7.6)$$

which returns us to the model (6.2).

Although formally, definition (7.5) coincides with the ordinary definition of the velocity function under the assumption of continuity (an infinite number of microscopic events between e_j and e_{j+1}), the latter is subject to application only in the case when both of the following claims are justifiable:

- knowledge of the "exact" Hamiltonian;
- knowledge of the initial conditions with "infinite precision".

[h] To simplify the notations, numeric indexes near τ and h are omitted.
[i] It cannot degenerate into a point due to the existence of the macro-level.

Neither of these two can be guaranteed even for a simplified dynamic motion [44,45]. Whereas in the classical definition of the velocity function we relate microscopic points in the macroscopic frame of reference, (7.5) establishes a correspondence between two macroscopic events on the probabilistic basis of microscopic events between them. Hence, *the GDS velocity is a measure of changes which take place on the microscopic level with respect to the macroscopic behaviour of the system.* If we assume that such changes are vanishing, $\lim_{\epsilon\to 0^+} v_\epsilon = v_1$, then we can expect (see (7.6)) that

$$\lim_{\epsilon\to 0^+, n\to\infty} v(t,x) = v_1. \tag{7.7}$$

We call the mathematical idealization of evolution described by the model (6.2) with the limiting velocity defined by (7.7) an Infinite Length Unperturbed Markov Chain (ILUMC). The reality of perturbations ($\epsilon > 0$) implies an approximation $v_\epsilon \approx v_1$ that leads to the computational idealization of an Infinite Length Perturbed Markov Chain (ILPMC). The approximate relationship

$$\lim_{n\to\infty} v(t,x) \approx v_1 \tag{7.8}$$

reflects our endeavours to describe the evolution of PGDS. In general, mathematical modelling of GDS evolution according to (7.8) implies an approximation of the macroscopic velocity function with respect to an inevitable approximation of the function of micro-velocity. Such an approximation can be seen as the choice of a countable base in a topological space that induces a transformation from a space–time of events of PGDS to a discrete space–time of macroscopic events of this system evolution. This assumes a passage from the grid of macroscopic events $\omega_{h\tau}$ defined by (7.4) to a new grid, nodes of which are computational models of these events defined by a topology base in the macroscopic frame of reference.

A consideration of the *space–time as a causal discrete set* was the subject of many publications (see, for example, [7,9] and references therein). Recently some new theoretical results on dynamic system discretizations on lattices have been obtained [14]. Below we formalize these ideas with respect to our models using the Markov Chain approach. First, the state space of the initial macroscopic event e_0 has to be specified with respect to

- absolute time of the decision maker, and
- an approximation of the system–environment boundary at the initial moment of such absolute time.

In the case of a one-dimensional approximation we define this space in the macroscopic frame of reference as

$$\Xi(i;0) = \{x_i, i = 0,1,2,3,\ldots,N; \quad N = 2n,\ n = \lceil (T - t_0)/h \rceil\}. \tag{7.9}$$

We assign to each state of Ξ a particular probability weight p_i^0, which can be defined on the basis of the micro-velocity approximation with the property of decreasing probabilities $1 \geq p_0^0 > p_1^0 > p_2^0 \cdots > p_N^0 \geq 0$ (the theoretical limit of "infinite precision" is not excluded). Thus, to define the state space of a macroscopic event, we include a theoretical possibility of GDS evolution in each cell of the grid of macroscopic events. If $h_i = h$, $i = 0, 1, \ldots, n-1$, then $\max_j \tau_j \leq h$ and the limiting case of equality leads to a consideration of a square grid $\omega_{\tau h}^{\square}(m = n)$ which has the resolution to identify any macroscopic event relevant to system evolution in $\bar{G}$ when $n \to \infty$. This case implies $h \to 0$ (and as a consequence $\tau \to 0$) when the state space of the initial macroscopic event defined according to (7.10) degenerates into a ray that indicates the loss of connection between absolute DM-time and relative time of the dynamic system. We can circumvent this *problem of uncontrolled propagation of initial uncertainty* by a probabilistic description of macroscopic states which are subject to conservation of the Markov condition on the basis of an appropriately constructed Markov Chain associated with GDS evolution.

DEFINITION 7.2 A set of macroscopic events defined by a mapping $\omega_{\tau h}^{\square} \to \Xi(i;j)$, where

$$\Xi(i;t^j) = \{(x_i, t^j),\ i = \overline{k, 2n-k},\ j = k,\ k = \overline{0,n}\}, \tag{7.10}$$

is called the *cone of macroscopic events of system evolution.*

Remark 7.2 The formula (7.10) in Definition 7.2 is given for a "one point target in absolute DM-time" and can be generalized for any target set including a set of isolated points in the DM-time scale ($t = T$). This may be of the great importance for some optimal control problems.

Our next step is an approximation of the macro-velocity function with respect to the micro-velocity using Definition 7.1. As a characteristic of the microscopic velocity function $\epsilon > 0$ we use a numerical index $\chi_n = o(\tau + h)$ defined in the macroscopic frame of reference by probability weights of the neighbourhood states of an associated Markov Chain.

DEFINITION 7.3 A Markov Chain $\xi_n^{\tau h}$, $n < \infty$ is *consistent* with the Markov process $(h(\tau), \mu(t, x))$ defined by the mathematical model of GDS evolution (7.1), (7.2) if

$$E_n^{\tau h|(x_i,\mu^j)} \Delta\xi_j^{\tau h} = v_1(x_i, t^j, \mu^j)\tau + o(h + \tau) \quad (7.11)$$

and

$$\operatorname{cov}_n^{\tau h|(x_i,\mu^j)} \Delta\xi_j^{\tau h} = o(h + \tau) \quad (7.12)$$

hold. We refer to the condition (7.11) as *the condition of local consistency*, whereas (7.12) is referred to as *the global consistency condition.*

Remark 7.3 The equalities (7.11), (7.12) imply the fact that the macroscopic properties of the system should not change dramatically in small (with respect to the whole evolution) DM-time-sets, although microscopic properties can vary significantly subject to the velocity function. Another way of putting it is that consistency conditions referring to the probabilistic microscopic level make explicit basic features of system evolution on the macroscopic level. The same role in physics is played by the second law of thermodynamics [45].

In general, even if in the reality of dynamic system evolution there exists

- a uniform movement of the microscopic frame of reference with respect to the macroscopic one with a velocity v_e, and
- a linear dependency of the corresponding points (x, t) and (τ, h),

these facts can be established neither by mathematical modelling nor by a measuring experiment. However, the limiting case of our consideration (when $h \to 0$ and hence $\tau \to 0$) implies that

$$\operatorname{cov}_n^{\tau h|(x_i,u^j)} \Delta\xi_j^{\tau h} \to 0 \text{ when } n \to \infty.$$

Of course, the infinite length Markov Chain is within the scope of the Markov theorem on the generalized law of big numbers.

THEOREM 7.1 *If a sequence of arbitrary random values* $\Delta\xi_1, \Delta\xi_2, \ldots, \Delta\xi_n, \ldots$ *satisfies the condition*

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} \operatorname{cov}[\Delta\xi_i] = 0,$$

then the limiting result

$$\lim_{n\to\infty} P\left\{ \left| \frac{1}{n} \sum_{i=1}^{n} (\Delta\xi_i - E[\Delta\xi_i]) \right| \geq \varepsilon \right\} = 0$$

holds for any arbitrary $\epsilon > 0$.

Therefore, if we construct a Finite Length Perturbed Markov Chain (FLPMC) with the properties (7.11) and (7.12), we can guarantee convergence of such an approximation to ILPMC in the probabilistic sense of Theorem 7.1 when the number of macroscopic states $n \to \infty$. The limit passages

$$\begin{aligned} \Delta\xi_n^{\epsilon} = & \Delta\xi_n^{h\tau} \text{ (if } n \to \infty \text{ then)} \\ & \to \Delta\xi_{\infty}^{\epsilon} \text{ (if } \epsilon \to 0 \text{ then)} \to \Delta\xi_{\infty}^{00} \end{aligned}$$

illustrate schematically a connection between FLPMC, ILPMC and ILUMC. An approximation error of FLPMC with respect to ILUMC is defined by

$$E(\epsilon, n) = \delta(\epsilon) + \Delta_n,$$

which vanishes in the limit $\epsilon \to 0^+$ and $n \to \infty$. In this case[j] the macro-velocity of the system coincides (see (7.7)) with the velocity of the associated ILUMC, and

$$(\xi_n^{\tau h}, \mu(x_i, t^j)) \to (h(\tau), \mu(x, t)).$$

Any other cases assume a probabilistic description of physical states (see [44]) that can be associated with an appropriately constructed Markov Chain. It makes it necessary to transform the continuous space–time of a macroscopic frame of reference into the discrete space–time of macroscopic events of system evolution, that is to construct the cone of macroscopic events. The base of this cone is subject to the implementation of the complementarity principle (6.7), which acknowledges the fact of the system existence at the initial moment of DM-time with the probability 1[k]. We note that as an alternative approach there is the theoretical possibility to control possible changes of macro-velocity from the micro-level. In general, using an appropriate approximation (that is valid for the macroscopic level of system description), we can describe the event e_0 in the two complementary forms

- either position-and-DM formulation as $(x_0, 1)$,
- or time-and-macro-velocity formulation as $(t_0, 0)$.

Theoretically, we can combine both approaches by considering the problem in terms of macro-velocity and the DM-function that corresponds to the specification of the event e_0 as (0,1). Such a consideration is typical for mathematical models in optimal control theory, where the decision maker plays the role of the "error-nulling" optimizing device of a modeller type. This approach can be regarded *as the velocity–control formulation of evolutionary problems.* An alternative consideration of initial conditions as $(1, 0)$ seems to be intrinsic to the investigation of biological self-organizing dynamic systems. DM in such cases can be associated with the "observer", and this approach can be formally regarded as *the velocity–energy formulation of evolutionary problems.* To combine both possibilities in such a specification of the event e_0, computational models of dynamic system evolution should be derived. The main difficulty that immediately arises stems from the necessity of an approximation of the limit of $n\epsilon(n)$ for any dynamic system which evolves in space–time ($n \to \infty$) under the possibility of vanishing perturbations ($\epsilon \to 0^+$). The method proposed in this paper is based on such a construction of computational event-models in the cone of macroscopic events that preserve the stability property of associated evolution. In general, such an approach permits the DM to switch from "observer" to "modeller" and vice versa whenever it is necessary.

To construct a stable approximation of the model (7.1), (7.2) the idea of the upwind discrete scheme with flux limiters [57] is used. Without loss of generality for the numerical procedure, we assume that $\tilde{f}_0 = 0$, which reduces Eq. (7.1) to 6.6(b). First, let us introduce in the cone of macroscopic events (7.10) a floating grid:

$$\omega_{\tau h}^{\Delta} = \{(x_i, t_j^{\tau_{j-1}}, \quad i = \overline{k, 2n-k}, \ j = k, \ k = \overline{0, n}\}, \tag{7.13}$$

where $t_j^{\tau_{j-1}} = t^{j-1} + \tau_{j-1}$ when $j > 1$, $t_j^{\tau_{j-1}} = t^0 + \tau$ when $j = 1$, and $t_j^{\tau_{j-1}} = t^0$ when $j = 0$. Provided all τ_{j-1}, $j = \overline{1, n}$, $\tau_0 = \tau$ are defined, the grid (7.13) generates a set of approximations to the macroscopic events defined by $\Xi(i; j)$. Since for a particular DM-time $t_j^{\tau_{j-1}}$ an associated event depends only on the macroscopic event that corresponds to the t^{j-1}-moment of DM-time, the value of τ_{j-1} is subject to stability conditions for the system. Such conditions depend on the velocity of the system, which is approximated using an evolution-associated Markov Chain. Now if we

[j] When classical concept of continuous phase space trajectories can be formally applied.
[k] However, it does not give a way to specify the initial condition for the macro-model (7.1), (7.2).

denote approximations to μ-function and v_1 on $\omega_{\tau h}^{\Delta}$ as d and v respectively, then the approximations

$$\frac{\partial \mu}{\partial x} \approx \begin{cases} (d^+_{i+\frac{1}{2}} - d^+_{i-\frac{1}{2}})/h & \text{if } v_i < 0, \\ (d^-_{i+\frac{1}{2}} - d^-_{i-\frac{1}{2}})/h & \text{if } v_i > 0 \end{cases}$$

allow us to derive the discrete scheme

$$d_i^j = d_i^{j+1} + \tau \left\{ \frac{d^-_{i+\frac{1}{2}} - d^-_{i-\frac{1}{2}}}{h} v^+ - \frac{d^+_{i+\frac{1}{2}} - d^+_{i-\frac{1}{2}}}{h} v^- \right\}, \tag{7.14}$$

where $v^+ = \max[v_i, 0]$, $v^- = \max[-v_i, 0]$ and

$$d^-_{i+\frac{1}{2}} = d_i^j + \Delta d_i^j \gamma_1(r_i), \quad d^-_{i-\frac{1}{2}} = d^j_{i-1} + \nabla d^j_{i-1} \gamma_2(r_{i-1}),$$
$$d^+_{i+\frac{1}{2}} = d^j_{i+1} + \Delta d^j_{i+1} \gamma_3(r_{i+1}), \quad d^+_{i-\frac{1}{2}} = d_i^j + \nabla d_i^j \gamma_4(r_i).$$

Here γ_i, $i = \overline{1,4}$ are flux limiters which are subject to definition with respect to the velocity function approximation. The other notations are the common

$$\nabla d_i^j = d_i^j - d^j_{i-1}, \quad \Delta d_i^j = d^j_{i+1} - d_i^j, \quad r_i = \frac{\nabla d_i}{\Delta d_i}.$$

Then the discrete scheme (7.14) can be rewritten in the form

$$\begin{aligned} d_i^{j+1} = {} & d_i^j \{1 - \frac{\tau}{h}[|v| + v^- \gamma_4 - v^+ \gamma_1]\} \\ & + \frac{\tau}{h} d^j_{i-1} \{[v^+(1+\gamma_2) + v^- \gamma_4]\} \\ & + \frac{\tau}{h} d^j_{i+1} \{[v^-(1-\gamma_3) - v^+ \gamma_1]\} \\ & + \frac{\tau}{h} d^j_{i-2} \{[-v^+ \gamma_2]\} + \frac{\tau}{h} d^j_{i+2} \{v^- \gamma_3\}. \end{aligned} \tag{7.15}$$

A verification of the sum of all coefficients near unknown function on the right hand side of (7.15) gives unity. Hence, provided nonnegativeness conditions are satisfied, we can associate these coefficients with transition probabilities of a Markov Chain. In fact, the conditions of nonnegativeness of probabilities are

$$1 - \frac{\tau}{h}(|v| + v^- \gamma_4 - v^+ \gamma_1) \geq 0, \quad \gamma_2 \leq 0, \quad \gamma_3 \geq 0, \tag{7.16}$$

$$v^+(1+\gamma_2) + v^- \gamma_4 \geq 0, \quad v^-(\gamma_3 - 1) + v^+ \gamma_1 \leq 0. \tag{7.17}$$

The partial cases of $v^- = 0$ $(v^+ \neq 0)$ and $v^+ = 0$ $(v^- \neq 0)$ give the results

$$1 - \frac{\tau}{h} v^+(1 - \gamma_1) \geq 0, \quad \gamma_1 \leq 0, \quad -1 \leq \gamma_2 \leq 0,$$

and

$$1 - \frac{\tau}{h} v^-(1 + \gamma_4) \geq 0, \quad \gamma_4 \geq 0, \quad 0 \leq \gamma_3 \leq 1,$$

respectively.

LEMMA 7.1 *Under the conditions* (7.16), (7.17) *the Markov Chain defined by time-transitions of the discrete scheme* (7.15) *is locally consistent with the process* $(h(\tau), \mu(t,x))$ *defined by the model* (7.1), (7.2) *if the equality for flux limiters*

$$\tau[v^-(1 - \gamma_4 + \gamma_3) - v^+(1 + \gamma_1 - \gamma_2) - v] = o(\tau + h) \tag{7.18}$$

holds.

Proof If a previous state of the Markov Chain was $\xi_j^{h\tau} = x$ subject to control d^j, then according to the assumption of Lemma 7.1 we have the following table of transition probabilities for a new state $\xi_{j+1}^{h\tau}$:

New state	Probability of transition
$x - h$	$\tau/h[v^+(1+\gamma_2) + v^- \gamma_4]$
$x + h$	$\tau/h[v^-(1 - \gamma_3 - v^+ \gamma_1]$
x	$1 - \tau/h[\lvert v\rvert + v^- \gamma_4 - v^+ \gamma_1]$
$x - 2h$	$\tau/h[-v^+ \gamma_2]$
$x + 2h$	$\tau/h[v^- \gamma_3]$

Therefore it can be verified that

$$\begin{aligned} E_j^{\tau h|(x,d^j)} \Delta \xi_j^{\tau h} = {} & -\tau[v^+(1+\gamma_2) + v^- \gamma_4] \\ & + \tau[v^-(1-\gamma_3) - v^+ \gamma_1] \\ & + 0[1 - \frac{\tau}{h}\{|v| + v^- \gamma_4 - v^+ \gamma_1\}] \\ & - 2\tau[-v^+ \gamma_2] + 2\tau[v^- \gamma_3] \\ = {} & \tau[v^-(1 - \gamma_4 + \gamma_3) \\ & - v^+(1 + \gamma_1 - \gamma_2)]. \end{aligned}$$

This equality together with definition (7.11) completes the proof.

Remark 7.4 The Markov Chain velocity $v_{\mathrm{MC}} = [v^-(1-\gamma_4+\gamma_3) - v^+(1+\gamma_1-\gamma_2)]$ between two successive macroscopic events coincides with the velocity of the process when $n \to \infty$. For any finite value of n we have $v_{\mathrm{MC}} \geq v$ which corresponds to the nonnegativeness of the covariance of the Markov Chain jump between these macroscopic events.

LEMMA 7.2 *Under the conditions* (7.16), (7.17) *the Markov Chain defined by time-transitions of the discrete scheme* (7.15) *is globally consistent with the process* $(h(\tau), \mu(t,x))$ *if the equality for flux limiters*

$$\tau\{h[v^+(1-\gamma_1-3\gamma_2)+v^-(1+\gamma_4+3\gamma_3)] - \tau v_{MC}^2\} = o(\tau+h) \tag{7.19}$$

holds.

Proof In a way similar to what was done in the proof of Lemma 7.1, we construct the following table of transition probabilities:

New state	Value of $[\Delta\xi - E\Delta\xi]^2$
$x-h$	$(-h-\tau v_{\mathrm{MC}})^2$
$x+h$	$(h-\tau v_{\mathrm{MC}})^2$
x	$(0-\tau v_{\mathrm{MC}})^2$
$x-2h$	$(-2h-\tau v_{\mathrm{MC}})^2$
$x+2h$	$(2h-\tau v_{\mathrm{MC}})^2$

We notice that the probabilities of transitions correspond to those from the transition probability table in Lemma 7.1. Therefore the computation of covariance

$$\begin{aligned}
&\mathrm{cov}_j^{\tau h|(x,d^j)}\Delta\xi_j^{\tau h} \\
&= E[\Delta\xi - E\Delta\xi]^2 \\
&= \frac{\tau}{h}\Big\{[h^2 + 2h\tau v_{\mathrm{MC}} + \tau^2 v_{\mathrm{MC}}^2] \\
&\quad \times [v^+(1+\gamma_2) + v^-\gamma_4] \\
&\quad + [h^2 - 2h\tau v_{\mathrm{MC}} + \tau^2 v_{\mathrm{MC}}^2] \\
&\quad \times [v^-(1-\gamma_3) - v^+\gamma_1] \\
&\quad - \tau^2 v_{\mathrm{MC}}^2\{1 - \frac{\tau}{h}[|v| + v^-\gamma_4 - v^+\gamma_1]\Big\} \\
&\quad + [4h^2 + 4h\tau v_{\mathrm{MC}} + \tau^2 v_{\mathrm{MC}}^2][-v^+\gamma_2] \\
&\quad \times [4h^2 - 4h\tau v_{\mathrm{MC}} + \tau^2 v_{\mathrm{MC}}^2][v^-\gamma_3]\} + \tau^2 v_{\mathrm{MC}}^2 \\
&= \tau h[v^+(1-\gamma_1-3\gamma_2) + v^-(1+\gamma_4+3\gamma_3)] \\
&\quad - \tau v_{\mathrm{MC}}^2,
\end{aligned}$$

gives the required equality (7.19), if we take into account (7.12).

Remark 7.5 For each cell $c_{ij} \subset \omega_{\tau h}$ a probabilistic analogue of the characteristics of Eq. (7.1) can be defined by the equality

$$cov_j^{\tau h|(x,d^n)}\Delta\xi_n^{\tau h} \pm \tau v_{\mathrm{MC}} = const. \tag{7.20}$$

To estimate the value of *const* in (7.20) we can eliminate the term $o(\tau+h)$ in our approximation using (7.11) and (7.12):

$$cov_j^{\tau h|(x,d^j)}\Delta\xi_j^{\tau h} - \tau v_{\mathrm{MC}} = -\tau v. \tag{7.21}$$

Using Lemma 7.2 the equality (7.21) can be rewritten as

$$h[v^+(1-\gamma_1-3\gamma_2) + v^-(1+\gamma_4+3\gamma_3)] - \tau v_{\mathrm{MC}}^2 = \tau(v_{\mathrm{MC}} - v). \tag{7.22}$$

Therefore nonnegativeness of covariance is equivalent to the stability condition

$$\frac{\tau}{h} \leq \frac{v^+(1-\gamma_1-3\gamma_2) + v^-(1+\gamma_4+3\gamma_3)}{[v^-(1+\gamma_3-\gamma_4) - v^+(1+\gamma_1-\gamma_2)]^2}, \tag{7.23}$$

which follows directly from (7.22). Provided flux limiters are chosen in such a way that the equality

$$\begin{aligned}
&v^+(1-\gamma_1-3\gamma_2) + v^-(1+\gamma_4+3\gamma_3) \\
&\quad = [v^-(1+\gamma_3-\gamma_4) - v^+(1+\gamma_1-\gamma_2)]^2
\end{aligned} \tag{7.24}$$

holds, the stability condition (7.23) is satisfied.

Example 7.1 Examples of the choices of flux limiters are given below for two partial cases.

- If $v^- = 0$ and $\gamma_2 = 0$ $(i=j)$ then the value of the flux limiter γ_1 can be found from (4.16) in the form

$$\gamma_1 = -1 - \frac{\sqrt{8v^+ + 1} + 1}{2v^+}.$$

- If $v^+ = 0$ and $\gamma_3 = 0$ $(i = N-j)$ then the value of the flux limiter γ_4 is defined as

$$\gamma_4 = 1 + \frac{\sqrt{8v^- + 1} + 1}{2v^-}.$$

The identification of flux limiters completes the construction of the discrete scheme which defines the Markov Chain with the corresponding interpolation interval τ (subject to stability conditions) and transition probabilities. We state the result in the form of the theorem on the Markov-Chain-approximation stability in discrete space–time of events.

THEOREM 7.2 *If transition probabilities of a Markov Chain* $(\xi_n^{\tau h},\ n < \infty)$ *are defined by the formula*

$$p^{\tau h}[x_k^j, x_i^{j+1} | d(x_k^j, t^j)] = \begin{cases} 1 - \frac{\tau}{h}[|v| + v^-\gamma_4 - v^+\gamma_1], & k = i, \\ \frac{\tau}{h}[v^+(1+\gamma_2) + v^-\gamma_4], & k = i-1, \\ \frac{\tau}{h}[v^-(1-\gamma_3) - v^+\gamma_1], & k = i+1, \\ -\frac{\tau}{h}(v^+\gamma_2), & k = i-2, \\ \frac{\tau}{h}(v^-\gamma_3), & k = i+2, \\ 0, & \text{otherwise}, \end{cases}$$

$\forall j = \overline{0, n-1}$ *and* $i = \overline{j, N-j}$ $(\gamma_2 = 0$ *for* $i = j$ *and* $\gamma_3 = 0$ *for* $i = N-j)$, *whereas the interpolation interval* τ *satisfies the conditions* (7.16), (7.17), (7.23), *then the Markov-Chain-approximation of the process* $(h(\tau), \mu(x, t))$ *is stable, and discrete values of the DM-function can be found from the formula*

$$d(x_i^{j+1}, t^{j+1}) = \sum_k p^{\tau h}[x_k^j, x_i^{j+1} | d(x_k^j, t^j)] d(x_k^j, t^j). \quad (7.25)$$

Remark 7.6 (*on convergence*). When $n \to \infty$ the velocity of the Markov-Chain converges to the velocity of the process in the sense of Theorem 7.1. If we consider, for example, a formulation of the problem in terms of velocity-control, then due to the complementarity principle the discrete function (7.25) converges to the decision maker function of the system.

Remark 7.7 (*on numerical procedures*). A numerical method proposed in this section is an explicit (evolution forward) stabilization procedure where the DM-function is a stabilizing factor subject to the velocity of the system.

Remark 7.8 (*on backward evolution operators and continuity of phase space trajectories*). A probabilistic description of event e_{n_0} precludes the situation where terminating data for backward evolution procedures can be specified in a "deterministic" way. Moreover, states $x(t_0)$ and $x(T)$ of the system in DM-absolute-time scale can be characterised by different probability weights, which makes the continuity assumption for the connecting trajectory inapplicable in general.

8 COMPUTATIONAL ASPECTS OF DISCRETE MARKOV DECISION PROCESSES

In a vicinity of any event e_0 which we might conditionally associate with the *present* of GDS evolution, there are infinitely many events relevant to the GDS evolution which might be called *past* and *future* events of evolution. As a result, an event itself can be formalized mathematically, neither with a deterministic certainty, nor with a precise probability. This implies difficulty in justifying the separability of topological spaces when the evolution of UGDS and PGDS is investigated.

Let us denote a probabilistic error of the inevitable approximation of such an event in the initial conditions of a mathematical model as $\varrho_0^\epsilon \in (0, 1]$, $\epsilon > 0$. Then the principal mathematical assumption which allows us to develop analytical theory of dynamic system in continuous (space-) time is *a possibilistic assumption of vanishing error*

$$\lim_{\epsilon \to 0^+} \varrho_0^\epsilon = 0. \quad (8.1)$$

Moreover, a concept of absolute or "external" to the system DM-time [45,38] leads to the theoretical possibility of predicting a future event e_{n_0} which is

associated with the DM-time $t=T$ (possibly $T=\infty$) with the probability 1. This means that

$$\lim_{\epsilon\to 0^+} \varrho^{\epsilon}_{n_0} = 0, \tag{8.2}$$

where $\varrho^{\epsilon}_{n_0}$ is a probabilistic error in the definition of this event. This approach (which is deterministic in its essence) usually visualizes evolution as a continuous trajectory $x(t)$ between present e_0 (time $t=t_0$) and future e_{n_0} (time $t=T$) events along which positions of the system can be determined at least in principle with the probability 1. Assuming that (8.2) holds, let us try to go backward in continuous DM-time. If evolution of the system in continuous space–time has taken place at all, we can select between events e_{n_0} and e_0 at least (n_0-1) events relevant to system evolution, which we will refer to as macroscopic. Further, we can extract between macroscopic events e_1 and e_0 at least (n_1-1) events relevant to system evolution which we will call microscopic, and will denote as $e^{01}_1, e^{01}_2, \ldots, e^{01}_{n_1-1}$. In the same way, we can find (n_2-1) sub-microscopic events $e^{011}_1, e^{011}_2, \ldots, e^{011}_{n_2-1}$ etc. As a result, we obtain a functional of the event-transition-error in the form

$$\begin{aligned} F(x,t) = &\sum_{i=1}^{n_0-1} \varrho_{i+1}(\varrho_i) + \sum_{i=1}^{n_1-1} \varrho_{i+1,0}(\varrho_{i,0}) \\ &+ \sum_{i=1}^{n_2-1} \varrho_{i+1,00}(\varrho_{i,00}) + \cdots \\ &+ \sum_{i=1}^{n_k-1} \varrho_{i+1,\underbrace{00\ldots0}_{k}}(\varrho_{i,\underbrace{00\ldots0}_{k}}) + \cdots, \end{aligned} \tag{8.3}$$

where, for example, a probabilistic error in a transition between events e^{011}_i and e^{011}_{i+1} $(i=1,\ldots,n_2-1)$ is defined by $\varrho_{i+1,00}(\varrho_{i,00})$. To guarantee convergence of the series in the right-hand side of (8.3) we should require

$$\lim_{k\to\infty} \varrho_{2,\underbrace{00\ldots0}_{k}}(\varrho_{1,\underbrace{00\ldots0}_{k}}) = 0,$$

where, assuming that (8.1) holds, we also have

$$\lim_{k\to\infty} \varrho_{1,\underbrace{00\ldots0}_{k}} = 0.$$

Applying the same arguments in the forward DM-time we can draw the conclusion that for any "middle" macroscopic event $e_m \in (e_0, e_{n_0})$ (DM-time $t_m \in (t_0, T)$) both events e_0 (DM-time $t=t_0$) and e_{n_0} (DM-time $t=T$) are *infinitely far from it in the continuous (space-) time of events*. However, in the macroscopic frame of reference, the distance between the events e_m and e_0 as well as a distance between e_m and e_{n_0} are well-defined in terms of absolute DM-time by the intervals $\Delta_{0,m}=t_m-t_0$ and $\Delta_{m,n_0}=T-t_m$ respectively. In other words, provided that both assumptions (8.1) and (8.2) can be justified, any event e_m of GDS evolution has two time-characteristics: (absolute macroscopic) DM-time $t_m \in (t_0, T)$ and (relative microscopic) system-time $\tau_m \in (-\infty,+\infty)$. The mathematical formalism, that allows us to circumvent the arising difficulty of time scaling, is based on the Cauchy-type models, and requires an exact specification of initial (or terminal) conditions for the position-vector or the density function in a separable topological space. Eventually, mathematically rigorous justification of such models requires simultaneous application of the concept of a time-infinity (either in the form of ergodic-type hypotheses or infinite-step algorithm) and the possibility of vanishing perturbations when time goes by. Another way of putting it is that infinite time is a *necessary* condition for the justification of unperturbed mathematical models. However, *sufficiency* of this condition is subject to possibility theory [15,49] rather than the theory of probabilities. From the physical point of view the analysis of the described problem requires the concept of relative time. The mathematical idealization which reconciles the concepts of absolute and relative time of dynamic system evolution is ILUMC in the continuous space–time of events, for which the claim of $(t,\tau) \in (-\infty,+\infty)$ is natural. The very next step in the modelling of dynamic system evolution is ILPMC. Such models imply an *approximation* of an event e_0 that formally gives two rays in relative-time directions $((-\infty,\tau_0)$ and $(\tau_0,+\infty))$. Our knowledge of the relative time τ_0 is based on its intermediate influence on the quality

of approximations of objects of mathematical modelling with respect to the moment t_0 of absolute time. A selection of one of the two rays in relative-time directions corresponds to the choice of a Markov semigroup [44] associated either with a covariance-non-negative (for future) or a covariance-non-positive (for past) Markov Chain. Whatever model is chosen, the Markovian property for the evolution should be preserved by an appropriate *algorithm*. It requires consideration of perturbed mathematical models with the specified level of error.

An approximation of event e_0 implies a truncation of the series $F(x, t)$ in (8.3). Let us denote a probabilistic error induced by such a truncation as

$$\varrho_0^\epsilon = \varrho_{1,\underbrace{000}_{l}} > 0.$$

Let us also assume that the limit of vanishing error,

$$\lim_{l\to\infty} \varrho_{1,\underbrace{00\ldots0}_{l}} = \lim_{\epsilon\to 0} \varrho_0^\epsilon = 0,$$

holds. Then, in general, the quality of prediction of GDS evolution by means of mathematical modelling is defined by the quality of a solution of the optimization problem

$$\sum_{i=0}^{\infty} \varrho_{i+1}^\epsilon(\varrho_i^\epsilon) \to \min. \tag{8.4}$$

Since the difference between an unperturbed trajectory x_t of ILUMC and a perturbed trajectory x_t^ϵ of ILPMC at a certain moment $t = t_m$ of DM-time can be arbitrary big, the necessary condition for convergence of series (8.4),

$$\lim_{n\to\infty} \varrho_{n+1}^\epsilon(\varrho_n^\epsilon) = 0,$$

cannot be guaranteed in general, no matter how small $\epsilon > 0$ is assumed. This is not a surprising fact since in general the optimizing function is a function of an infinite degree of recursion of the density function. The intrinsic idea in mathematical modelling and computational experiments is to reduce the degree of recursion to a finite number. In doing so we arrive at the problem

$$\sum_{i=0}^{n_0} \tilde{\varrho}_{i+1}^\epsilon(\tilde{\varrho}_i^\epsilon) \to \min,$$

which implies the construction of FLPMC. Though the difference between two macroscopic states x_k^ϵ and x_{k+1}^ϵ in DM-time scale might still be arbitrary big in general (between two macroscopic events e_k and e_{k+1} there might be an infinite number of microscopic events relevant to system evolution), we are now able to estimate a probability of corresponding transition using the values of $\tilde{\varrho}_{i+1}^\epsilon(\tilde{\varrho}_i^\epsilon)$. By means of FLPMC we preserve the stability of the macroscopic system (the object of mathematical modelling) with respect to its microscopic dynamics. Although stability of the microscopic dynamics with respect to a macroscopic system will follow in the limit of our construction, any finite time computational procedure is not necessarily a reflection (even qualitatively) of the latter. To put it differently, using tools of mathematical modelling, results generated by ILUMC or ILPMC (i.e. a complete description of GDS evolution) cannot be guaranteed with the probability 1. If it is granted that mathematical modelling can give a way to describe the real processes, systems, and phenomena, then a conceptually necessary passage from continuous trajectories ($x(t)$ or $x^\epsilon(t)$) in absolute ("external" to the system) DM-time to a probabilistic description of physical states should be undertaken. A convenient framework for a probabilistic description of system evolution from one macroscopic event to another provides the concept of DMDP [26]. Since DMDP is considered in the macroscopic frame of reference, both

- a number of observed macroscopic events (which is finite $n_0 = n_0(T, t_0, h(\tau))$), and
- a topology of the state space,[1]

depend on an approximation of initial e_0 and terminating e_{n_0} events. In the macroscopic frame of

[1] which can change in general with respect to absolute DM-time due to fluctuating system-environment boundaries.

reference the state space gives rise to the cone of macroscopic events

$$\Xi(i;j) = \{x_i^j,\ i = \overline{n_j, N_j}\},$$

where for $j=0$ we have $n_j=0$ and $N_j=N$. In the case of 1-D approximation (one-point-target), the cone of macroscopic events was defined by (7.10). At a certain moment t^j of absolute DM-time, the system can be in one of the states x_i^j to which we assign different probabilistic weights p_i^j. With the same probability weights we associate the corresponding action set defined by

$$\mathcal{A}(i;j) = \{\mu(t^j, x_i^j),\ i = \overline{n_j, N_j}\}.$$

If we now define an allowable decision set for each macroscopic event $e_j, j = \overline{0, n_0}$ as

$$\mathcal{D}(i;j) = \Xi(i;j) \otimes \mathcal{A}(i;j),$$

then the construction of a probabilistic model for each macroscopic event of system evolution

$$e_j \equiv \{\mathcal{D}(i;j);\ p_i^j\}$$

has been completed. If $\mu(t^j, x_i^j) \forall i = \overline{n_j, N_j}$ is known, a description of the macroscopic event at DM-time t^j becomes totally deterministic. A reward set is defined by the probability distribution of the next macroscopic event:

$$R(e_j \to e_{j+1}) = (\{(x_i^{j+1}, \mu(t^{j+1}, x_i^{j+1}); p_i^{j+1}\},\ i = \overline{n_{j+1}, N_{j+1}}).$$

The following stabilization procedure (which is described with respect to the approximations used in Section 7) can be applied as an implementation of Theorerm 7.1.

ALGORITHM 8.1

- *Initialization of initial event.*
  Find initial values of the DM-function at $t=t^0$. *That is, define event* e_0 *by triples* $(x_i^0, \mu(x_i^0, t^0); p_i^0)$. *Then set complementary description of the initial event as* $(t^0, v_i^0; p_i^j)$.
- *Prediction step for an event-model.*
  Given values of $\mu(x_i^0, t_0)$, *define an approximation to the velocity function of the process* $v(t^0 + h, x_i + h, \mu(x_i^0, t^0)) = v_i^1, i = \overline{2, 2n}$ *at the next DM-time-layer* $t^1 = t^0 + h$. *Set* (t^1, v_i^1) *as an approximate description of the next macroscopic event.*
- *Correction step for the event-model.*
  Using the approximate description of the event in terms of time-velocity, find flux limiters and the time-step of stability τ, *for which define an event-model as* $(t^{1,\tau}, \mu(x_i^{1,\tau}, t^{1,\tau}; p_i^1)$ *where* $i = \overline{2, 2n}$ *and* $t^{1,\tau} = t^0 + \tau$.
- *Event definition.*
  The definition of DM-function by $\mu(x_i^1, t^1) = \mu(x_i^{1,\tau}, t^{1,\tau})\ \ i = \overline{2, 2n}$ *gives the new macroscopic event in the form of the set of triples* $(x_i^1, \mu(x_i^1, t^1); p_i^1)\ i = \overline{2, 2n}$.
- *Complementary description of the event.*
  Define complementary description of the event as $(t^1, v_i^1; p_i^1)$ *and repeat the procedure for the next DM-time-layer, etc.*

Since in the DM-time scale (where the stabilization procedure has to be employed) a real event always follows after its event-model counterpart, this implies that an error at each step of the procedure is defined by a time-discrepancy between the event and its event-model (for example, the first step of 1-D approximation gives $\Delta t = t^1 - t^{1,\tau} = h - \tau$). To minimize this error we should find a Markov strategy which at each DM-time level chooses the highest probability of a transition. In general we have the whole family of DMDP defined as

$$\mathcal{M}(e_0, e_1, \ldots, e_{n_0}) = \{\mathcal{D}(i;j), R(e_j \to e_{j+1}),\ p_j^{j+1}(k;i),\ i = \overline{n_j, N_j},\ j = \overline{0, n_0}\}, \tag{8.5}$$

where $p_j^{j+1}(k;i) = p^{h\tau}[x_k^j, x_i^{j+1} | \mu(x_k^j, t^j)]$. Each of such a DMDP constructs a *probabilistic trajectory of system evolution*

$$\mathcal{T}(k_0, \ldots, k_{n_0}) = \{p_0^0(k_0), p_0^1(k_0; k_1), p_1^2(k_1; k_2), \ldots, p_{n_0-1}^{n_0}(k_{n_0-1}; k_{n_0})\},$$

($k_l \in [n_l, N_l]$, $l = \overline{0, n_0}$), in the cone of macroscopic events. In general the equality $p^{n_0}_{n_0-1}(k_{n_0-1}; k_{n_0}) = 1$ cannot be guaranteed, and closeness of this probability to 1 depends on values of $p_0(k_0)$ and the structure of the cone of macroscopic events (i.e. on the approximation of e_0 and e_{n_0}). To single out amongst all probabilistic trajectories defined by DMDP (5.5) an optimal one we define the probabilities of successful prediction as

$$\bar{p}^0_0(k^*_0) = \max_i p^0_0(i; i), \quad i = \overline{n_0, N_0},$$

and then

$$p^1(k^*_0; i) = \max_k p^1_0(k; i),$$
$$\bar{p}^1_0(k^*_0; k^*_1) = \max_i p^1(k_0; i), \quad i = \overline{n_1, N_1}.$$

In general we have

$$p^{j+1}(k^*_j; i) = \max_k p^{j+1}_j(k; i),$$
$$\bar{p}^{j+1}_j(k^*_j; k^*_{j+1}) = \max_i p^{j+1}(k_j; i),$$

where $i = \overline{n_j, N_j}$, $j = \overline{1, n_0 - 1}$. Then the policy

$$\pi = \{(x_{k^*_l}, \mu(x_{k^*_l}, t^l),\ l = \overline{0, n_0}\},$$

induced by *the optimal probabilistic trajectory*

$$\mathcal{T}^*(k^*_0, \ldots, k^*_{n_0}) = \{\bar{p}^0_0(k^*_0), \bar{p}^1_0(k^*_0; k^*_1), \bar{p}^2_1(k^*_1; k^*_2), \ldots, \bar{p}^{n_0}_{n_0-1}(k^*_{n_0-1}; k^*_{n_0})\},$$

we call *the optimal Markov policy*, which gives (in the DM-time scale) the Markov Chain approximation to GDS evolution $(h(\tau), \mu(t,x))$. The DMDP, which is associated with this policy, corresponds to the construction of such a Markov Chain which evolves to the most probable state of the system preserving of strong causality of macroscopic events.

9 CONCLUSION

In this paper mathematical modelling of dynamic system evolution has been studied as a problem in information theory. Computational models for evolution based on the ideas of evolution-associated Markov Chain approximations have been developed. Since the velocity function of the system is coupled to perturbations of its environment, stability conditions for the system have been derived in an explicit form.

Mathematical models for the evolution of dynamic systems are closely connected with discrete optimization problems through the definition of information and the associated notion of entropy for thermodynamic systems. Information uncertainty in knowledge bases influences the construction of mathematical models, and should be taken into account. This implies a certain heuristic nature in such a construction. Such heuristic approaches are an important part of studying dynamic system evolution, and will remain as such in the foreseeable future, supplementing achievements obtained with the increasing computational power of modern computers and improved methods of data collection and analysis. Moreover, hybrid procedures combining the features of constructive, sequential, and evolutionary algorithms of discrete optimization give a general framework that could challenge well-established techniques in optimization theory.

Many important breakthroughs in optimization theory are intrinsically connected with the application of algorithms of sequential analysis that are based on the Markovian-type schemes. Such schemes are typical in computational models where minimax concepts of optimality are used. A mathematical formalization of the problem is quite natural, and is computationally consistent. The problem is viewed as attempts by the decision maker to obtain the best guaranteed result with respect to available information about the problem. The same formalization is a starting point for constructing mathematical models where other (such as probabilistic) concepts of optimality are used. In applications of such decision-maker schemes there is a natural contradiction between a desire for informational completeness in the model that is being constructed and a desire to

choose functional classes for which effective computational algorithm exists. In a search for a compromise between these two extremes induced by the "energetic" (combinatorial) and informational complexity of the underlying algorithm [56,58] it is reasonable to include the decision maker as an intrinsic part of the constructed model using some learning rules. As a result, mathematical models become coupled to their computational associate. This allows us to look for the optimal algorithms as those that at each step of their performance in the best way to use the information, which is accumulated by this step. The number of steps and quality of performance can be mathematically defined by the degree of recursion of an approximation to the system Hamiltonian (with respect to the density function) and the parameter of perturbations. In studying dynamic system evolution it is expected that a compromise between the two mentioned types of complexity can be achieved by the requirement of system stability. This cannot be guaranteed in general unless the underlying model is defined by *hyperbolic* rather than purely *parabolic* dynamic rules. Examples of this type have been derived, and the limiting cases of vanishing perturbations and infinite recursion rule have been discussed. The results on the derivation of hyperbolic equations of the Hamilton–Jacobi–Bellman type for non-smooth and stochastic optimal control will be published separately [39,40,41]. Their connection with the principles of extended irreversible thermodynamics [46,30] as well as computational algorithms shall be also discussed elsewhere.

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