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Mathematical Models for Climate as a Link between Coupled Physical Processes and Computational Decoupling

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Mathematical models for climate studies are treated as a coupling link between physical and computational models. These models are characterized by the fact that small-scale phenomena influence the large-scale properties of the modelling system. Yet the former cannot be extracted from the latter using available hardware and computational procedures. Climate systems belong to the class of systems whose dynamics are only observable in transient states. As a result, the sensitivity of models to coupling procedures requires an examination of the schemes responsible for transporting data between components. It is proposed to perform such an examination, based on the connection between error growth and the degree of coupling of model components, using adaptive error control.

K e y w o r d s : Mathematical climate system models, coupling and decoupling procedures, hydrodynamic stability.

*1. **Introduction**.* Elements of the mathematical modelling of climate can be traced back to Aristotle's Meteorologica. The first contemporary achievement in this field is often attributed to U. Leverrier, who was the first to produce a weather map after a big storm in France in November 1854. A rigorous modern basis for short-range weather prediction was laid by Vilhelm and Jacob Bjerknes and other scientists of the Bergen school. For the first time they rigorously treated the problem of weather forecasting as a mechanico-mathematical problem. Such a problem was described mathematically by an initial value problem for the hydrodynamic equations of a baroclinic fluid. One of the main concepts introduced in this work was *the concept of wave instability on frontal interfaces,* which is still of primary importance in theoretical meteorology.

An important step in mathematical climate studies was made by L. Richardson and Courant, Friedrichs, Lewy in the field of numerical analysis. The results of the latter group gave a guideline for the explanation of some failures in the former work, where meteorological "noises" (such as acoustic waves) were included in the model.

During the early 1940s through to the 60s, many scientists contributed to filtering noise from the solution for the hydrodynamic system. In the field of theoretical meteorology a number of pioneering works were published, establishing basic principles for simplification of the hydrodynamic equations through the quasi-geostrophic expansion. Mathematical foundations of the theory were laid using a probabilistic approach [22, 39]. The main difficulty with the early works in this direction was the formulation of boundary conditions for the problem. Alternatives to the geostrophic approximations of the hydrodynamic equations were also developed. Among them was the quasi-solenoidal approximation.

Serious mathematical difficulties in the practical application of quasi-geostrophic and quasi-solenoidal approximations led in the late 1950s to a return to the initial hydrodynamic equations which were essentially used by L. Richardson. Such equations in theoretical meteorology are called primitive. Since that time, the main developments have been concentrated on numerical methods and the improvement of models by a better physical parameterization. Many important factors related to the physical parameterization were taken into consideration and implemented into models. This led to the creation of modern state-of-the-art models for climate that consist of relatively independent components that are responsible for interconnected parts of climate such as atmosphere, ocean, land surface, sea ice, etc, and require substantial computational power to obtain approximate solutions.

The quality of these approximate solutions depend significantly on the consistency between the mathematical model and the real climate. Since the improvement of mathematical models can be achieved by improved physical parameterization, *the concept of coupling between different components* becomes straightforward.

From the physical point of view, climate studies are essentially based on three fundamental theories. These are thermodynamics, the theory of radiation, and magnetohydrodynamics. Since a description of the climate system should include both the earth and the atmosphere, the overall system is often referred to as the atmosphere-active-layers (AAL) system. Both the earth and the atmosphere require a detailed physical parameterization, that leads to the difficulty of mathematically formalising *the interaction between underlying processes and phenomena*. The main difficulty with the description of the earth is to measure a purely gravitational force. In fact, we can only observe the combined effect of the two forces, gravitational and centrifugal, that is referred to as gravity. The difficulties in measurements stem from

- the variation of gravity at different latitudes;
- the variation of gravity vertically with respect to sea level;
- the variation of gravity erratically with respect to the earth's crust and other irregularities.

The main difficulty in the description of the atmosphere is to adequately represent transport effects in models. Amongst the most important constituents in the atmosphere are water (about 4% per volume), carbon dioxide (about 0.03% per volume), ozone (about $0.1 \cdot 10^{-5}$ % per volume), oxygen (about 20%) and nitrogen (about 70%). However, if water and carbon dioxide are present throughout the atmosphere, then ozone becomes influential only at 20-30 km from the earth's surface, oxygen only from about 80 km and nitrogen even higher [37]. In addition, many gaseous constituents only have an indirect influence through the propagation of electromagnetic waves. There are also many important nongaseous constituents such as condensed water, salt particles, dust and others that play important roles in the description of clouds and precipitation.

We conclude that transport phenomena and gravity are key factors for an adequate description of the AAL system.

2. The structure of the paper and notation.

This paper is organised as follows:

- Section 3 gives a brief outline of space-time scales that are important in a climate study. We recall the main hypothesis that is used in the mathematical modelling of synoptic processes.
- In Section 4 we consider the distinction between short and long range predictions on the basis of the concept of the relaxation time. The fundamental equations for the adiabatic approximation are also presented in this section.
- Section 5 is devoted to non-adiabatic models and their simplifications on the basis of quasi-geostrophic and quasi-solenoidal approximations.
- In Section 6 we consider advantages in the return to the primitive hydrodynamic equations, and difficulties in an adequate representation of the vertical structure of meteorological fields.
- Section 7 deals with non-adiabatic factors that lead to an approximation of the conservation law in mathematical models.
- Sections 8—10 are devoted to a dilemma between the concepts of coupling and independence as well as approaches for the numerical treatment of an interplay between these two concepts.
- In Sections II and 12 we formulate a finite set of differential equations that provides an approximation to the dynamics of climatic processes. We address two questions related to such an approximation, namely phase transitions and algorithmic stability.
- Section 13 deals with questions related to the validation of mathematical models for climate. We argue that for the validation of models, both a-priori assumptions and a-posteriori information are needed.
- In Section 14 we present some numerical results of the computation of meteorological fields on the basis of the NCAR CCM3 model.
- Section 15 concludes the paper. Directions for future development are also presented in this section.

The following notation is used throughout the paper:

- g is the acceleration due to gravity;
- ρ is the density;
- p is the pressure;
- T is the temperature;
- z is the altitude;
- m is the mass of the earth;
- c_0 is the isothermal speed of sound;
- h_a is the height of the atmosphere;
- p_0 is the average surface pressure;
- ρ_0 is the average surface density;
- T_0 is the average air temperature at sea level;
- k_b is the Boltzmann constant;
- k_v is the von Karman constant;
- $k_r = c_p/c_v$ is the ratio of specific heat capacity under constant pressure to that at constant volume;
- R is the gas constant, for example, $R = 287\, Jkg^{-1}K^{-1}$ for dry air and $R = 461\, Jkg^{-1}K^{-1}$ for water vapour;
- $\Omega = 7.292 \times 10^{-5} s^{-1}$ is the angular velocity of the earth;
- φ is the latitude;
- $\chi = 2\,\Omega \sin\varphi$ is the Coriolis parameter;
- $\boldsymbol{F} = (F_x, F_y, F_z)$ is the field external to the earth, excluding pressure-gradient forces;
- c_p is the specific heat capacity at constant pressure, for example, $c_p = 1. \times 10^3\, Jkg^{-1}K^{-1}$ for dry air and $c_p = 1.81 \times 10^3\, Jkg^{-1}K^{-1}$ for water vapour.

Other notation is explained in the text as required.

3. Space-time scales and their interaction.

In climate studies one of the most challenging problems is to adequately describe space-time scale interactions [23]. By interacting between themselves, different bio-chemico-physical processes at different scales form a unified whole which we call climate. The problem of such interactions is typically simplified mathematically by regarding microturbulence as a dissipative factor which can be characterized by an *effective (or dynamic) viscosity coefficient*. Such a simplification allows us to effectively model synoptic oscillations, i.e. climate processes that are characterized by time scales from hours to several days. Diurnal oscillations also belong to this class. Amongst other types of oscillations, the following classes can be distinguished:

- Global oscillations, for example, planetary oscillations. They play an essential role in long-term weather predictions. Their time scales are characterized by periods from weeks to months. The Atmospheric Boundary Layer is a key factor in such processes.

- Seasonal oscillations that vary over a year.
- Interannual oscillations with time scales of several years. To this class belong, for example, glacial periods and ENSO-type phenomena.
- Micrometeorological oscillations with time scales of seconds to minutes. Small-scale turbulence, acoustic waves, and gravitational waves with small amplitudes provide examples of this type of oscillations.
- Mesometeorological oscillations such as thunderstorms, and gravitational waves with large amplitudes. They typically last from minutes to an hour.

We emphasize that mathematical models of climate systems are essentially "proxy" climate systems. Whatever model is chosen, small scale phenomena may substantially influence large-scale properties of the system, but computational procedures may not be available to extract the former from the latter.

4. Short and long range in the prediction of meteorological fields.

The short-range prediction of meteorological fields is based on hydrodynamic theory in the case where the energy of sources and sinks is virtually ignored by using the adiabatic approximation. In quite a general setting, the resulting equations can be derived from the two conservation law equations, the conservation of the entropy and the conservation of the "vortex charge" [37],

$$\frac{d\Xi}{dt} = 0, \frac{d\Psi}{dt} = 0 \tag{4.1}$$

where $\Psi = (\boldsymbol{\Psi}_0 \cdot \nabla \Xi)/\rho$ is the potential Rossby vorticity. Due to (4.1), the entropy function Ξ and the absolute vorticity $\boldsymbol{\Psi}_0$ are generators of differential adiabatic invariants, because any function of them is again an adiabatic invariant. As an integral invariant, all systems that can be described by the model (4.1) have the total energy of the system, E, constant.

In the general case, the well-posedness of the model (4.1) is not an established mathematical fact. Any specific choice of two independent Lagrangian coordinates as well as the definition of two parts of the integral invariant (for example, the kinetic energy and the labile energy that, in turn, consists of the sum of the potential energy and the internal energy of the system), implies the necessity of addressing the problem of system stability. E.N. Lorenz was the first who proposed addressing this problem using the macrostability parameter, S. Let

$$\Theta = T(p_0/p)^{(k_r - 1)/k_r} \tag{4.2}$$

be the potential temperature, where p_0 is the standard pressure. This quantity is often convenient as one of the Lagrangian coordinates. Then, the parameter of macrostability, S, can be defined as the weighted average value of the vertical gradient of the potential temperature over the entire thickness of the atmosphere. If K is the kinetic energy of the system, then the quantity $K - S$ is also on adiabatic

invariant. Hence, the quantity S shows the amount of kinetic energy that is released/absorbed in the process of adiabatic transitions. This approach to stability requires an adequate specification of the vertical structure of the atmosphere.

An alternative approach is based on *the concept of relaxation time*, τ. From the mechanical point of view, the relaxation time, or "build-in" period, can be seen as the atmospheric efficiency coefficient, i.e. the rate at which potential energy, E_p, is converted into kinetic energy:

$$\tau = \left(\frac{1}{E_p} \frac{\partial E_p}{\partial t} \right)^{-1}. \tag{4.3}$$

On the scale of synoptic processes, $\tau \approx 1$ week. If the time interval of interest $t - t_0$ (where t_0 is an intial moment of time) is less than τ, the model (4.1) may provide a good approximation for short-range weather changes. However, for periods that satisfy the inequality

$$t - t_0 > \tau \tag{4.4}$$

practically all regions of the atmosphere have sufficient time to interact with each other and the model (4.1) becomes inappropriate. Since the atmosphere is a rapidly changing component with low inertia of the whole AAL system, an essential part of investigation in the field of climate study is being concentrated on atmospheric modelling. This approach gives rise to the major difficulty in the modelling of long-term meteorological changes. We have to fix the initial state of the whole physical system, namely when $t = t_0$. Of course, this requires more careful examination of other components of the system, in particular the ocean, which is a component with a large thermal inertia. Hence, one of the most important initial conditions for the whole model is the temperature field. It is well-known, for example, that incompleteness of such data causes problems in modelling processes such as ENSO phenomenon, and other processes in the equatorial zones where the Coriolis parameter vanishes and the structure of the boundary layer of the atmosphere has to be modelled with an increased precision [16, 62, 6, 33, 60]. The implementation of *non-local features of the system* into the model becomes important for the validity of the model [13]. In the end, this requires an appropriate description of turbulence in the boundary layer as a major factor responsible for the mixing of heat, momentum, passive scalars, moisture etc. This emphasises the importance of taking into account both the interaction of time-scales and the interaction of spatial scales [48].

As we mentioned in the introduction, three main types of physical processes, namely

- thermodynamic,
- radiative, and
- magnetohydrodynamic,

influence the output of mathematical models of climate subjected to the physical parameterization [29, 45]. As a result, many efforts during recent times have been

concentrated on improvements of existing physical parameterizations. In particular, much attention has been devoted to an adequate modelling of radiative processes [4] that require appropriate models for cloudiness and the transport of tracer species [47]. The analysis of sensitivity to transport phenomena has led to an increased interest in the semi-Lagrangian approach as an alternative to the well-established spectral approaches [46]. The semi-Lagrangian approach requires special numerical procedures for interpolation to compensate for the sparse character of data.

The necessity of an adequate representation of transport phenomena in mathematical models naturally leads to the development of the concept of coupling with respect to different components of climate [5]. In addition to the large spatial scale that cover thousands of kilometers other scales, such as

- mesoscale (from kilometres to hundreds of kilometres),
- small scale (from dozens of metres to kilometres), and
- microscales (from millimetres to dozens of metres).

become important for long-range prediction. Of course, in climate applications smaller scale structures are inevitably described statistically. It does not follow, however, that such structures are necessarily random in nature. The only a-priori conclusion we can draw is that in long-term processes *the atmosphere as a whole does not act as a closed system* [37]. It acts as a component of a bigger AAL system composed of the atmosphere and active layers that can be described by a coupling of many different physical, chemical and biological fields.

5. Physical hypotheses and mathematical approximations.

In long-term prediction, equations for the conservation of entropy and the potential vorticity cannot provide an adequate description of underlying processes that are essentially nonadiabatic. Since in this case we should be able to adequately describe sources and dissipation of energy, certain assumptions about the laws of dissipation and accumulation of energy should be made. In this case, model (4.1) should be replaced by evolutionary equations that more adequately represent transport phenomena. Let us denote by Λ the rate of energy increase per unit mass, and by $\boldsymbol{F}_v$ the viscous force per unit mass. Then such equations can be written in the form

$$T\frac{d\Xi}{dt} = \Lambda, \rho\frac{d\Psi}{dt} = \operatorname{div}[(\Lambda/T)\,\Psi_0 + \Xi(\operatorname{curl}\boldsymbol{F}_v)]. \tag{5.1}$$

This model allows us to take into account non-adiabatic effects. It introduces two new variables, momentum and heat fluxes, and requires additional hypotheses on sources and sinks in the AAL system. The central hypothesis is *the hypothesis of local thermodynamic equilibrium*. During recent years theoretical and experimental physicists proved it was necessary to go beyond the framework of this hypothesis [38, 35]. We note, that model (5.1) is a model of local type. In fact, to derive this

model, the Obukhov hypothesis on the conservative properties of the potential vorticity is used [37], and the field

$$(\Lambda/T)\,\Psi_0 + \Xi\,(\mathrm{curl}\,F_v) \tag{5.2}$$

is assumed to have a solenoidal structure. Nevertheless, in models of this type, difficulties connected with the inherently approximate modelling of physical processes such as the polarization of radiation, refraction, dispersion, and cloudiness, may still be partially overcome by introducing some feedback mechanisms, like regulators through the cloud cover, sea ice, snow cover etc.

The model (5.1) with a realistic physical parameterization is extremely difficult to deal with without some additional simplifications. Once again, we can use the local equilibrium hypothesis for mathematical simplifications of climate system models. This hypothesis leads to satisfactory results, at least in the case of small-amplitude waves. In this particular case it is sufficient to relate small oscillations of the atmosphere to the equilibrium state. When oscillations are small *the perturbation theory technique* is natural for the investigation of solutions of resulting models. In its essence, the successful application of this theory in many areas has its roots in the local equilibrium hypothesis.

Let us assume that in the equilibrium state pressure, density and temperature, p, ρ and T, depend only on the amplitude z. Then, as a primary task, we have to describe the dynamics of these variables near the equilibrium. In climate study, these variables are connected, as a rule, by the "timeless" hydrostatic (or quasi-static) equation,

$$\frac{\partial p}{\partial z} = -\rho g, \tag{5.3}$$

and the Clapeyron equation

$$p = \rho R T, \tag{5.4}$$

where, as usual, g is the acceleration due to gravity and R is the specific gas constant.

For now, we shall assume that at the initial moment of time t_0 the atmospheric motion is

- quasi-static, i.e.

$$\partial p_e / \partial z = -g\rho_e; \tag{5.5}$$

- horizontal, i.e. for the velocity field $\boldsymbol{v} = (u, v, w)$ we assume

$$w = 0; \text{ and} \tag{5.6}$$

- geostrophic i.e. the velocity field is assumed to be non-divergent,

$$u_e = -\frac{1}{\chi\rho}\frac{\partial p_e}{\partial y},\ v_e = -\frac{1}{\chi\rho}\frac{\partial p_e}{\partial x}, \tag{5.7}$$

where χ is the Coriolis parameter and the subscript index e stands for the values of thermodynamic parameters at moment t_0. Of course, in this case one can introduce the stream function by the formula

$$\psi = p_e / (\chi \rho), \tag{5.8}$$

and the formulas (5.7) may be rewritten as

$$u_e = -\frac{\partial \psi}{\partial y}, \quad v_e = \frac{\partial \psi}{\partial x}. \tag{5.9}$$

One of the assumptions that is often made is that these three properties (referred to as consistency conditions) will be preserved in the future. This guarantees a stationary solution for the set of equations governing the atmosphere. Such motion is called motion of the first kind or *slow motion*. By the standard technique we can also account for the curvature of the earth by the transformation of the stationary solutions into slow gyroscopic Rossby waves.

If consistency conditions are violated in any region of space, *X*, *fast motion,* or motion of the second kind, has to be taken into consideration. In reality, we observe a continuous competition between a violation of the consistency conditions and adaptation of the meteorological fields $\boldsymbol{v}, p, \rho, T$. Since traditional methods assume that "meteorological noise" has little significance in the weather prediction, numerous attempts have been made to filter out motions of the second kind. However, if we accept the quasi-static approximation, all frequencies of the internal acoustic waves go to infinity ("complete filtering"). Frequencies of the gravity waves become overestimated, though the error decreases for longer waves.

The weather is typically associated with synoptic processes. Hence, in order to describe such processes at a "minimal cost", we have to filter out from solutions of the Eulerian hydrodynamic equations (they contain both slow and fast motions) the motions of the second kind. If it is assumed that only the motion of the first kind is important for synoptic processes, then in the Eulerian equations we have two dimensional parameters, g and χ, that can be used for such filtering. Nonlinear systems in which fast oscillations occur along with slow ones are typical in applications (probably, the most widely cited example is the Van-der-Pol equation for the description of oscillations in an electric circuit containing a vacuum tube with feedback). Asymptotic methods and the theory of perturbations were developed in order to mathematically treat such systems. The idea of expansion with respect to a small parameter can also be used as a simplification of the hydrodynamic system to make it appropriate for the description of synoptic processes.

Let L and U be typical length and speed scales for synoptic processes. Then the role of small parameters may be played by the Rossby-Kibel number

$$R_k = U/(\chi L), \tag{5.10}$$

or the Mach number

$$M_a = U/c_0. \tag{5.11}$$

The isothermal sound speed, c_0, is defined using the height of the atmosphere, h_a, as

$$c_0 = \sqrt{g h_a}, \text{ where } h_a = p_0/(\rho_0 g) \tag{5.12}$$

The Rossby-Kibel number may be interpreted as the ratio of the typical relative acceleration, U^2/L, to the typical Coriolis acceleration χU [37]. Using standard series expansion, the consistency conditions for the first kind motion may be defined from conditions for the vertical potential vorticity,

$$\Psi_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{g}{f}\nabla^2 z + O\left(\frac{UR_k}{L}\right), \tag{5.13}$$

and for the horizontal divergence,

$$D_h = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = O\left(\frac{UR_k}{L}\right). \tag{5.14}$$

This expansion made with respect to the parameter UR_k/L is referred to as the quasi-geostrophic expansion. The quantities p_0 and ρ_0 in (5.12) are defined at the surface of the earth, and have to be included in the boundary conditions of the problem. We also need the initial field of the pressure to compute the pressure at future moments of time. However, the advantage of the quasi-geostrophic approximation is that formally we are not required to know the initial distribution of the velocity field. It should be noted that, strictly speaking, the resulting system can only be represented in the form of differential equations for synoptic processes in a barotropic-atmosphere approximation. A weak formulation of the problem is required for the more general case.

The quasi-geostrophic approximation is one of the possible approximations to conservation laws for the horizontal motion. In spite of its partial advantages it has serious drawbacks. The major one is a violation of the assumption of smallness for the Rossby-Kibel number, R_k, in the vicinity of the equator, since the Coriolis parameter $\chi = 2\Omega \sin\varphi$ decreases near the equator. Moreover, there is evidence that this approximation may be inadequate even outside of the tropical zones. In such cases the consistency conditions (5.13), (5.14) for the synoptic fields must be

changed. For example, in many cases it is reasonable to assume that the horizontal divergence D_h is small compared to the vertical vorticity Ψ_z, where

$$\Psi_z = O(U/L), \tag{5.15}$$

$$D_h = O\left(\frac{U}{L}\frac{M_a^2}{\alpha_0^2}\right). \tag{5.16}$$

In (5.16), α_0 is the parameter of (hydro)static stability. It is defined as

$$\alpha_0^2 = -(T/T_0)(p/c_p)\,\partial\Xi/\partial p, \tag{5.17}$$

and has an obvious connection with the Richardson number [37]

$$R_i = \frac{T_0^2}{M_a^2 T^2}\alpha_0^2 \tag{5.18}$$

(see also the definition in [11]). These modified consistency conditions, (5.15), (5.16), are derived with the assumption that in the horizontal velocity field for the slow (synoptic) motion, the potential component is small compared to the solenoidal component. Such an assumption leads to the quasi-solenoidal approximation. The mathematical model that corresponds to this approximation is typically formulated in terms of the stream function ψ, where

$$u = -\frac{\partial\psi}{\partial y}, v = -\frac{\partial\psi}{\partial x}. \tag{5.19}$$

The model requires both

- an initial field ψ for a given z, and
- values of ψ on the boundary.

It is well-known that the balance equation written for ψ is the Monge-Ampere-type equation that should be solved in a given bounded region. The well-posedness of such a problem in the general setting can be guaranteed when it is elliptical. However, the ellipticity condition [37],

$$g\nabla^2 z + \chi^2/2 > 0, \tag{5.20}$$

can only be satisfied when the quantity R_k is small. As we mentioned above, this condition is often violated in applications. Such a violation is most noticeable in the tropical zones.

In addition to the mathematical difficulties in using the quasi-solenoidal approximation, and unsatisfactory practical results obtained in many cases with the

quasi-geostrophic approximation, there remains the problem of filtering fast waves from the hydrodynamic equations open. However, progress in computational software and hardware has led to the possibility of solving the complete set of equations governing the atmosphere.

6. Primitive equations and the vertical structure of meteorological fields.

Since in many cases neither quasi-geostrophic nor quasi-solenoidal approximations are appropriate in applications, computational complexity is caused by filtering procedures. The development of numerical procedures from the original hydrodynamic system in these cases is no more complex. The main problem in modelling with the primitive equations is the formulation of appropriate boundary conditions. If boundary conditions are not properly formulated, the stability of the solution cannot be guaranteed. In fact, we need

- the normal velocity on the entire boundary, and
- the potential vorticity on the part of the boundary where air motion is directed toward the interior.

Because of the approximate character of available data for these boundary conditions, the formulation of the mathematical model requires two types of equations, prognostic equations and tendency equations. With the quasi-static approximation, the derivation of the tendency equation is straightforward, provided the vertical structure of meteorological processes is known. In the particular case of barotropic atmosphere, the problem of the vertical structure of synoptic processes is reasonably simple. This explains the early development of the theory in the direction of simplifications described in Section 5.

However, in the general case of a baroclinic atmosphere, the problem of the vertical structure of meteorological fields remains one of the most serious problems for mathematical modelling using the primitive equations. At the initial stage of development of the model, mainly pressure was used as the vertical coordinate. The implementation of the earth's orography [41] led to the adoption of a σ-system coordinate for the model. Among the first to use the geometrical altitude as a vertical coordinate was L. Richardson [50]. This idea was developed further by V. Starr (see references in [20]) who introduced quasi-Lagrangian coordinate systems. The present development of the vertical structure of meteorological fields in the NCAR CCM3 model [1] is based on the works of Kasahara and Washington [19], Kasahara [20], and Simmons and Stufing (see references in [21]). Among other approaches to vertical coordinates we include initial attempts at using the potential temperature (see (4.2)). This very fruitful idea has not received a proper development in the literature due to difficulties connected with lower boundary conditions.

From the mathematical point of view, the primitive equations are characterised by *the "restoration" of the hyperbolic operator in the model.* From the physical point of view we retain gravitational waves among the solutions. As a result, on the one hand

this approach requires a large number of initial data. On the other hand, the computational complexity of the resulting approximate solutions depends on values of small parameters such as the Rossby-Kibel number, R_k. We recall that this number characterizes the ratio between the inertial force of the system and the Coriolis force. Hence, if R_k is large (for example, when Ω is small or when L is relatively small), the Coriolis effect related to small-scale effects may be neglected, as is usually done in our everyday life, in spite of the rotation of the earth. However, if R_k is small, the complexity of approximating algorithms increases.

For many meteorological and oceanographic phenomena it is important to take into account the dynamic nature of such "small" parameters induced by the interaction of different space-time scales. At present, such an interaction is modelled on the basis of the classical law of viscosity that gives a connection between the stress σ and the speed $|\boldsymbol{v}|$ through the effective viscosity coefficient μ.

$$\sigma = \mu \frac{\partial |\boldsymbol{v}|}{\partial z}. \tag{6.1}$$

For practical applications, (6.1) should be supplemented by the law of energy dissipation. When microturbulence is treated as a dissipative factor, we can always find a reasonable analogy between the motion of molecules and the motion of macroscopic elements of turbulent fluids. This idea was first used by Prandtl [42], who attempted to treat the case of turbulent momentum exchange in this way. He arrived at the mixing-length hypothesis (see, for example, [12])

$$K_v = l^2 \left| \frac{\partial \boldsymbol{v}}{\partial z} \right|, \tag{6.2}$$

that gives a connection between the coefficient of vertical diffusion K_v and the velocity field $\boldsymbol{v}$ through the mixing length l. The connection between (6.1) and (6.2) should be provided by a scaling law. A major portion of current investigations in climate study is based on the scaling law of logarithmic type

$$\bar{u} = \frac{u^*}{k_v} \ln \frac{z}{z_0}, \tag{6.3}$$

where $\bar{u}$ denotes the mean velocity (in the x-direction), u^* is the friction velocity, z_0 is the roughness parameter, and k_v is the von Karman constant. Recently, new theoretical and experimental evidence was given to confirm that this law may be inappropriate as an adequate description of turbulent processes [2, 3].

7. *Long-term meteorological processes and non-adiabatic factors*

The main difference between short and long range meteorological processes is that in the long-range the atmosphere cannot be regarded as a closed system, since it

is a part of a bigger AAL system. Among the most important active layers is the ocean. After approximately 1—2 weeks the upper layer of the ocean has a substantial influence on atmospheric processes. As a result, one of the most important initial conditions in the model is the temperature field, in particular the temperature of the active layer of the ocean. At present, the practical availability of large datasets for such conditions has led to different ideas aimed at lengthening the period of validity of short-range models for the atmosphere. Initially, we can assume a constant temperature for the ocean, then use a slab-ocean model. Further, we can increase the number of formally independent mathematical models that can interact between themselves through message passing in a computational algorithm.

The main difficulty in the construction of long range models stems from the inadequateness of the adiabatic approximations. Conservation laws become approximate in nature, and one should take into consideration sources and dissipations of energy. In Section 5 we defined an approximation by model (5.1) that is based on the Obukhov hypothesis. Ultimately, the validity of approximations of this type is based on appropriate scaling laws. From the physical point of view, adequate construction of the model essentially depends on taking into account dissipative effects and sources, including

- heat sources such as solar and terrestrial radiation;
- cloud dispersion and absorbtion;
- local/nonlocal boundary layer diffusion etc.

The solution of mathematical models subjected to physical parameterizations that take into consideration such dissipative effects and sources can only be approached numerically. Moreover, the quality of the algorithm will decisively depend on the adequateness of the parameterization of bio-chemico-physical processes in the mathematical model. This is why, without additional simplifying assumptions with respect to non-atmospheric components of climate like ocean or sea ice, the "exactness" of conservation laws cannot be justified for any mathematical model. In practice, we always have to overcome difficulties arising from the approximate character of conservation laws in "proxy" climate models. Nevertheless, since the model is solved numerically, we can always use the idea of conservation on a finite grid [53, 5]. Indeed, the representation of conservation laws in mathematical models is of an approximate nature. For example, by a requirement on the vertical finite differences of the model to conserve the global integral of total energy in the absence of sources and sinks [5], we still neglect lack of conservation. In general, the stability conditions of the "proxy system" are not only different from those for the stability of the system itself, but they also are sensitively dependent on the degree of coupling achieved in the "proxy system". The approximate character of initial and boundary data in models with a hyperbolic-type operator does not allow consideration of the atmosphere in long-term processes as a closed system. Mathematical difficulties for such an approach are obvious. If the atmosphere is a

dependent component of the AAL system, then the questions *"how many such components are sufficient to adequately describe climatic processes"* and *"what are these other components"* have to be answered.

A wider mathematical "freedom" is allowed by looking at the evolution of states of the atmosphere as a random process $w(t)$. In this case it is possible to approach the task of studying the possibilities of the statistical extrapolation of this process using Kolmogorov's hypothesis. Namely, a random process $w(t)$ describing the evolution of turbulent flow in an environment with vanishing viscosity asymptotically approaches a Markov process for large t. From such a consideration it follows that the distribution of probabilities $P^t(dw)$ for $t > t_0$ may, in principle, be uniquely determined by the state $w(t_0)$, and not be dependent on the remote history of the process when $t < t_0$. Although the assumption of the negligible viscosity approximation can be justified on a finite grid (using, for example, the four-thirds Richardson's law), validation of the original mathematical model is intrinsically connected with the processing of incomplete information, which requires an adequate formulation of scaling laws. It appears that, in the general case, the Kolmogorov-Obukhov scaling law for local structures [22, 39], that is typically used, becomes inappropriate for this purpose. We discuss these issues in the next sections.

8. Information exchange between components of mathematical models for climate.

There are many "proxy climate" models which allow the simulation of interactions between different components by message passage in corresponding computational models. One of the models of this type is the NCAR CSM, where the original problem of climate study is reduced to that of a controlled exchange of in formation between the model components under the assumption that conservative properties (e.g. momentum, heat, freshwater) can be preserved when exchanged between model components [5]. In general, this assumption can be justified numerically, and it is reasonable to hope that by improving the physical (chemical, biological) parameterization we can improve the correctness and reliability of "proxy climate" models. Clearly, such improvements, as well as improvements in hardware and software, may continue indefinitely. Hence, it is necessary *to develop a strategy which permits an analysis of the trade-off between a level of coupling implied by achieved parameterization and a possible error induced by the incompleteness of available information.*

In climate study, an "exact" realization of conservation laws is closely connected with the consistency conditions for "slow" motion that implies hydrostatic, (5.5), horizontal, (5.6), and geostrophic, (5.7), approximations. In reality these conditions are continuously violated. As a result, approaches to filter out the fast waves from the solution of coupled system of PDEs are quite restricted in their applicability to climate study. Such approaches are typically based on the Kolmogorov-Obukhov scaling laws for local structures [22, 39] (we give more details in Section 10), and

more generally on the local equilibrium hypothesis [38]. It is often the case that statistical field theory can be used to give a practical explanation of these hypotheses. However, for many complex dynamic systems such as climate, mutual re-adjustment and self-adaptation of fields of different space-time scales is in the nature of the underlying processes. In such cases, the classical Kolmogorov-Obukhov scaling law may not lead to an adequate approximation of these processes [2, 3]. As we mentioned, the key point behind this fact is that the approximate character of initial data in models with a hyperbolic-type operator (such as the primitive equations of the hydrodynamic theory) *does not allow consideration of the atmosphere in long-term processes as a closed system.* Moreover, since the stability of any closed "proxy system" does not imply stability of the system itself, we need a trade-off strategy between coupling and stability concerns. To define such a strategy we have to appeal to the idea that a division between long-range and short-range depends on the definition of the relaxation time (or "build-up" period), τ (see (4.3)). This "coefficient of atmospheric efficiency" depends on the degree of coupling of atmosphere to its active layers that is ultimately defined by the problem solver or modeller. Mathematically, a formal division between "long" and "short" ranges is defined by the sign of the inequality between $t - t_0$ and τ :

$$r = \text{sign}\left\{ t - t_0, \tau \right\}. \tag{8.1}$$

The possibility of the existence of the two simultaneous limits

$$\tau \to 0^+ , \text{ and } t - t_0 \to \infty \tag{8.2}$$

is often taken for granted as an a-priori mathematical assumption in investigations of complex dynamic system. Mathematical models based on this assumption are characterised by a strong singularity at $\tau = 0$ and the parabolic features of the underlying dynamics. From the physical point of view such models are close associates of Fourier's original ideas on a diffusion mechanism for heat conduction. It is well-known, however, that the Fourier prediction may underestimate the peak temperature during a rapid transient period. Since experimental work on the wave behaviour of heat transport [40] and theoretical work in this field [58], interest in hyperbolic-type models for processes that include diffusion is being dramatically increased. In the general nonlinear case, such models preclude the assumption that small-scale phenomena can be extracted from a large-scale flow. As a result, one should overcome the problem of the approximate character of conservation laws in mathematical models of the "proxy systems". For example, heat in the "proxy climate system" might be conserved only under the assumption that it is neither gained nor lost at the top of the atmosphere. In fact, it is not conserved under inadequate parameterization (for example, if the long wave radiation in the atmosphere component uses the average sea surface temperature). Furthermore, in general the model can conserve energy only if we neglect the lack of conservation due to a-priori

regularity assumptions for our approximations. From the physical point of view, relaxation of these assumptions requires more careful examination of the non-local features of boundary layers [13]. We also note that in CCM-type models (we use a version of CCM in our computational experiment) the vertical advection of temperature is not used to conserve mass/energy. However, it is well known that the interaction between the vertical semi-Lagrangian approximations and the convective parameterization may seriously affect system predictability. In the general case, the a-priori assumption on the existence of conservation laws for "proxy" climate systems leads to a-priori regularity assumptions for "exact" solutions of mathematical models.

Since the construction of mathematical models for the evolution of thermodynamic systems has to be undertaken under analysis of uncertainty and the processing of incomplete information, it is always important to investigate the stability of associated computational models. However, in some cases it is possible to approach the issues of the well-posedness of the mathematical model without the investigation of stability in a computational sense. Indeed, mathematical analysis of the model itself can often be reduced to a general stochastic control problem. In turn, this problem can be often associated with a PDE of the Hamilton-Jacobi-Bellman type [9]. The quality of mathematical models based on such PDEs is essentially determined by the quality of the approximation of the system Hamiltonian, and approximate initial data for the models. In this case, the adequateness of the model to the real-world situation will be completely defined by the smoothness assumption on the sought-for solution. In the case where it is assumed that initial data for the model can be given exactly, the semi-continuity assumption [54] is natural. However, with application to real dynamic systems, such models can be reasonably validated if we are able to analyse the distinction between

- external error growth due to model deficiencies (such as physical parameterization), and
- the internal error growth due to mathematical assumptions (resulting from the unstable "self-growth" of the initial data errors).

The level of parameterization defines an upper bound for such an internal error. This provides a way to investigate the connection between coupling in mathematical models and the level of uncertainty in the model prediction (see [49, 8] and references therein). On the other hand, such a bound introduces hyperbolic features into the model [38].

9. Hybrid Eulerian-Lagrangian models and numerical schemes for transport effects.

Together with a wide use of the hybrid vertical coordinate, the interest in the hybrid Eulerian-Lagrangian type of mathematical models for climate study increases. In turn, this leads to attempts to implement into modern climate system models semi-Lagrangian advection approximations instead of the standard Eulerian approximations. One of the advantages of the semi-Lagrangian version is that in

many cases it allows us to relax the normal advective Courant-Fridrichs-Lewy (CFL) stability condition. In fact, it is well-known that for standard spectral models the typical resolution of the model may lead to instability (which may be observed in the Southern Hemisphere) if one applies a standard time-step. To obviate this problem, limited filtering is often used on the top model layer. Such models are often referred to as Eulerian with Spectral Transform (EST), though one realises that they are not Eulerian because the water vapor transport is usually treated in a semi-Lagrangian manner. Interest in the semi-Lagrangian type of models is stimulated by the claim (see [61] and references therein) that application of the semi-Lagrangian version may not only exclude the above-mentioned filtering, but also eliminate the normal advective CFL time-step restriction. However, it has to be recalled that the main problem with a semi-Lagrangian formulation consists of the fact that the result of interpolation with pointwise values is not a-priori conservative. As a result, long-term simulation can be seriously affected. We should admit that both semi-Lagrangian and EST versions have a serious deficiency. Small-scale features in the solutions may be underestimated more in semi-Lagrangian versions, whereas the EST approach has to deal with Gibbs phenomenon and spectral truncation. Hence, in practice, the EST methods can be successfully applied to adiabatic approximations, whereas the semi-Lagrangian approach is more natural for problems related to the advection of fields with large horizontal gradients (for example, when modelling water vapor transport). We should also take into consideration the fact that advantages in the stability of semi-Lagrangian advection schemes may be lost when we use hybrid Eulerian-Lagrangian models.

The author believes that the most efficient schemes must not require an explicit "subgrid-scale turbulence" parameterization and spacial filtering. Instead of the classical semi-Lagrangian or EST approaches it is reasonable to use 1D-Flux-Corrected-Transport (FCT) schemes. For climate study, this idea was used in [28]. The same idea was used in a different area of application to avoid explicit turbulence parameterization in [34]. It is straightforward to apply this idea to 3D transport modelling by means of time splitting in a manner explained, for example, in [32]. For climate models, zonal transport at high latitude can be split in time to satisfy the local CFL restriction, whereas at low latitude we can use a larger step. Such schemes provide an increasing accuracy with increasing resolution even when discontinuities or steep gradients are encountered. This property is extremely important to overcome the singular nature of the spherical coordinate system near poles, especially for systems with incomplete information, including those for which additional data from observations may be added in stages.

10. Turbulence, vanishing viscosity, and scaling laws.

The core of hydrodynamic theory is the system of the Navier-Stokes equations. The ensembles of solutions of these equations is usually associated with the behaviour of turbulence. Many theoretical works in this field are concentrated on the limiting case of vanishing viscosity in the Navier-Stokes system (the inviscid limit).

From the physical point of view, the system itself is a mathematical expression of conservation laws. Such laws are obtainable, at least in principle, from a hyperbolic system by adding a small viscosity coefficient. As follows from [25] (see also [26]), in the limit of vanishing viscosity one expects to be able to recover entropy solutions of the original hyperbolic system. The whole procedure is based on the assumption that a formal mathematical transformation from the hyperbolic system to the Navier-Stokes system preserves conservative properties. Naturally, if the original system is conservative, then after the vanishing-viscosity-limit transformation the system remains conservative. However, if we add to the original hyperbolic system a dispersive term, then we cannot expect that solutions of this modified system in the limit of vanishing dispersion are well-behaved. Existence or non-existence of solutions of such a modified system depend on the regularity of solutions of the original hyperbolic system. Therefore, though the Navier-Stokes system is parabolic in nature [38], in the general case it has both dispersive and hyperbolic features.

These features complicate the quantification of the behaviour of ensembles of the solutions of this system. Moreover, in the general case, the solutions of this system are non-stationary, and depend on the initial conditions for the model. From the theoretical point of view, not much can be said about the accuracy in the definition of initial conditions for complex dynamic systems. One of the standard approaches to the problem of investigating such systems is to start from stationary solutions, and try to approximate the time-average non-stationary solutions by averages of stationary statistical solutions. Such attempts are based on the ergodic hypothesis. From the physical point of view, such stationary random solutions are a generalization of steady state in the N-body problem. This generalization requires a well-defined concept of equilibrium. In classical theory, the equilibrium is associated with a "microcanonical distribution" obtained either by the equi-partition over the set of approximately equal energy systems, or by the Gibbs procedure. If the time-invariance of associated probabilities can be preserved, near-equilibrium processes can be investigated through equilibrium properties of the system using the approach introduced in the practice of mathematical modelling by Langevin (see, for example [14]).

If turbulence is regarded as a perturbation of an equilibrium, then more precise definitions of "equilibrium" and "perturbation" are required. If the equilibrium is obtained from the idea of the equi-partition of the set of approximately equal energy systems and the initial energy density is finite, then the resulting ensemble does not possess the ergodic property [2]. This type of equilibrium was first constructed in [15, 27] using the Fourier expansion. In the limit of vanishing truncation of the Fourier series, the underlying process can be described by a set of functions that are almost nowhere differentiable. From the physical point of view the process is characterised by the infinite energy density. If the topological space in which the set of these

functions can be embedded is a complete linear metric space, then under quite general assumptions [10] there exist no measures which are at least quasi-invariant relative to all translations, except for those identical to zero measure. Taking this into account, it is natural to regard real turbulent processes as those that are far from the Hopf-Lee equilibrium [2]. Mathematical investigations of such processes are intrinsically connected with the Fourier transform, and the theory of probability due to classic results of harmonic analysis (the Bochner theorem and its generalization to Banach spaces [44, 51]). At the core of such investigations from the physical point of view is the assumption that the provision of energy on a large scale is the dominant effect which can be modelled through the limit of vanishing viscosity on a small scale. There are two main types of mathematical models that are intrinsic to this approach:

- stationary models with exactly given boundary conditions;
- non-stationary models with exactly given initial conditions.

Both types of models have a common feature. Namely, nonlinear terms in the models can be treated as a perturbation expansion ordered by a small time-independent parameter. In climate system study the Rossby-Kibel number often plays the role of such a parameter, as we explained in Section 5. This approach encounters serious difficulty when the viscosity decreases and equilibrium is understood in the Hopf-Lee sense. In this case the singularity of turbulence increases simultaneously with the decrease in viscosity. Mathematically, this difficulty can be formally overcome by using the Kolmogorov-Obukhov scaling law [22, 39], which implies a certain character for energy distribution between scales. That is, an inertial cascade of energy from the "whirling" scales to the dissipation scales can be represented by the Kolmogorov spectrum [2]

$$E(k) = C\epsilon^{2/3}k^{-5/3}, \tag{10.1}$$

where k is the wave number, ϵ is the rate of energy transfer across the spectrum, and C is an absolute constant. Recent results on deviations from this law can be found, for example, in [2]. Such deviations may be expected when dispersion is intrinsic to the model. This is always the case for mathematical models of real phenomena or processes where physical (chemical or biological) parameterization cannot be performed with infinite precision. In the general case, it is not enough to consider the limit of vanishing viscosity in order to adequately describe turbulence. We also need information of the character of dispersion [26]. Such a correlation between viscosity and dispersion has stimulated the search for different principles on which the statistical theory should be built [2]. The other interpretation of turbulence which has been recently proposed is based on the assumption of a small perturbation of a suitable Gibbsian equilibrium. However, if we accept the Gibbs hypothesis (see [35] and references therein) the nature of the convergence of the probabilities in the limit of vanishing viscosity remains open. The answer to this question is to be found in the

approximation of the Hamiltonian. In fact, the Gibbs probability distribution is defined as the probability of a collection of states by the Lebesgue integral of

$$C_a = 1/n_0 \exp(-\beta H), \quad \beta = 1/k_b T, \tag{10.2}$$

with respect to the Liouville measure. In (10.2) the notation is standard, that is k_b is the Boltzmann constant, T is the temperature of the system (macroscopic temperature), n_0 is a normalizing factor and H is the system Hamiltonian. The definition of the system Hamiltonian is a hierarchically approximating procedure. This implies a procedure for obtaining conditions that single out the canonical ensemble measure from the class of all probability measures on the phase space of the system. Such conditions determine stability conditions of the model [36].

If the scaling law is agreed upon then *the nature of turbulence can be studied through different approximations of the system Hamiltonian*. The connection between the approximate character of the Hamiltonian and the scaling law can quantify turbulence numerically whenever the law of dispersion is established. As a result, the quality of mathematical models for turbulence is essentially determined by an adequate physical parameterization of the model that is linked to the hierarchical approximation of the system Hamiltonian.

11. Differential mathematical models for the climate study.

Many rigorous mathematical results in the investigation of climate models were obtained for the barotropic atmosphere. In this case, there is in the governing system of equations at least one equation that relates two thermodynamic variables on the basis of an individual particle from time to time. An alternative formulation can be given by using a piezotropic equation, which allows us to relate two thermodynamic variables from one spacial point to another at a given moment of time. Since statistical spacial data for climate study is typically sparse, the latter case leads to the weak rather than differential formulation of the problem. In general, nonlinear relationships between thermodynamic parameters which define the model suggest solution by numerical methods. Then, the quality of the underlying algorithm is completely determined by the correspondence of the model to the real climate.

We refer to the paper of J. Smagorinsky [52] on the basic experiment performed in collaboration with J.G. Charney, N.A. Phillips, and J. von Neumann. It is not reasonable to review here all subsequent steps in the development of the model that we use in this paper. The description is well-documented and the appropriate references related to the NCAR CCM3 model can be found in [21]. Instead, we give below the set of differential equations governing the atmosphere that is fundamental in climate study. It consists of five equations that relate the three components of wind $\mathbf{v} = (u, v, w)$, pressure p, density ρ and temperature T, namely

- the equations of motion

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} = 2\Omega(v\sin\varphi - w\cos\varphi) - \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \dfrac{F_x}{m}, \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} = -2\Omega u\sin\varphi - \dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \dfrac{F_y}{m}, \\ \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z} = 2\Omega u\cos\varphi - \dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g + \dfrac{F_z}{m}; \end{cases} \tag{11.1}$$

- the continuity equation

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = -\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right), \tag{11.2}$$

- the thermodynamic equation

$$d\varepsilon = c_p\, dT - \frac{1}{\rho}dp, \tag{11.3}$$

- and the state equation

$$\rho = \rho(p, T). \tag{11.4}$$

It is customary to use the Clapeyron equation $\rho = p/(RT)$ as the state equation. In this case the question of an adequate model of the latent heat of the phase transition is of primary importance because the gas constant R is different for different phases. In (11.3) $d\varepsilon$ is the heat added per unit mass, and the field of external forces in (11.1) is written excluding pressure-gradient forces (the pressure gradient is directed from high towards low pressure). In essence, equations (11.1)—(11.4) are prognostic. They provide a theoretical possibility for the mathematical forecast in climate study if we insert observed climatic conditions at a given time. A built-in physical parameterization splits the system (11.1)—(11.4) into prognostic equations and tendency equations.

Let us briefly recall typical assumptions made for simplifications of the system (11.1)—(11.4). The adiabatic approximation which we discussed in Section 4 can be obtained by setting $d\varepsilon = 0$, which provides a sufficient approximation for short range prediction. Moreover, if the atmosphere is assumed to be barotropic, the density at each point can be determined solely by the pressure at that point. In this case we have

$$\rho = \rho(p),\ T = T(p) \tag{11.5}$$

(the second relationship follows from the Clapeyron equation).

Furthermore, if the motion is assumed to be horizontal, we have

$$\begin{cases} \chi v = \dfrac{1}{\rho}\dfrac{\partial p}{\partial x}, \\ \chi u = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}, \\ g = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} + 2\,\Omega\, u \cos\varphi, \end{cases} \tag{11.6}$$

If we neglect the Coriolis term on the right-hand side of the last equation in (11.6), the equation turns into the (hydro)static equation. The first two equations of the system are geostrophic wind equations (or gradient balance equations) which may provide a good approximation in middle and high latitudes. This approximation simplifies numerical procedures because, for the barotropic atmosphere, geostrophic wind does not increase with height [11].

From the system (11.6) for the hydrostatic approximation pressure can be excluded. This results in the thermal wind equations

$$\begin{cases} \dfrac{\partial v}{\partial z} = \dfrac{g}{\chi T}\dfrac{\partial T}{\partial x} + \dfrac{v}{T}\dfrac{\partial T}{\partial z}, \\ \dfrac{\partial u}{\partial z} = -\dfrac{g}{\chi T}\dfrac{\partial T}{\partial y} + \dfrac{u}{T}\dfrac{\partial T}{\partial z}. \end{cases} \tag{11.7}$$

Again, numerically it leads to essential simplifications, since in the barotropic atmosphere the vertical temperature-gradient term is equal in absolute value and opposite in sign to the horizontal temperature gradient term. Of course, this gradient is not negligible in the general case.

In the baroclinic atmosphere a strong dependency between the horizontal temperature gradient and the vertical wind shear requires an *appropriate choice of the vertical coordinate.* The interdependency of different components of climate becomes important. However, the majority of implementations of the hydrological cycle into model (11.1)—(11.4), as well as the development of global general circulation models (that include in addition to the atmosphere other climate components such as ocean, land surface, sea ice), are essentially based on the hydrostatic equilibrium assumption [30, 31, 12]. The necessity for the development of a non-hydrostatic type of model has been realised during recent years [55].

Having included more than one climate components into a unified mathematical model, it becomes increasingly important to adequately formulate the hypothesis on subgrid scale vertical/horizontal mixing. Currently this hypothesis is formulated on the basis of the von-Karman-Prandtl logarithmic law (6.3) in the region of wall-bounded turbulent shear flow [59, 42, 43]. Since in general this law may lead to an inappropriate scaling [2, 3], more general laws for the interaction of space-time scales should be applied for the climate study.

12. Phase transitions and the algorithmic stability.

When temperature changes, water vapour departs from the ideal conditions, making an adequate model of water vapour transport one of the most important and difficult problem in climate study. The thermodynamics of water vapour and moist air is closely connected with the problem of latent heat and phase transitions. Even if the temperature of a substance remains constant, whenever this substance changes phase (evaporates, melts, condenses, freezes etc) a quantity of heat, called the latent heat of the phase change, must be supplied to or taken away from the substance. The quantification of the latent heat is based on the concept of entropy, and is often performed by using the Clausius-Clapeyron equation that relates the saturation of vapour pressure to the latent heat of a phase transition. Conceptually, this equation together with the state equation (11.4) is time-independent. This leads to an approximation of the mathematical model (11.1)—(11.4) whenever a physical parameterization is applied.

In most latitudes at most times of the year the atmospheric pressure and temperature vary continuously with time. As a result, *the geostrophic balance is never reached and maintained no matter how small the time-interval is assumed.* Rather, we observe a continuous re-adjustment of the fields with changing pressure and temperature fields. This requires the formulation of tendency equations that in turn requires some a-priori knowledge of the vertical structure of meteorological fields. On the other hand, the knowledge of the vertical structure of the meteorological processes is a major output in integrating the prognostic equations. Hence, although prognostic equations can provide a-posteriori information, they must always be supplemented by the tendency equations (which are based on a-priori information) in order to form *a closed system of mathematical equations.* The tendency equations are typically based on additional physical hypotheses (like hydrostaticity), and are approximate in their nature. The original system (11.1)—(11.4) is always replaced by its approximation, not only because of inevitable approximations of the functions F_x, F_y, F_z and initial and boundary conditions, but also because of the approximate nature of the equations (11.3) and (11.4) for any specific model. Naturally, this leads to attempts to improve such approximations by "building-in" to the model other components of climate such as the ocean, land-surface, sea ice. In these cases the vertical structure of meteorological processes cannot be defined in the simple manner of the barotropic approximation with the altitude typically defined by

$$z(x,y,t,p) = z_0(x,y,t)\,\psi_0(p). \tag{12.1}$$

On the other hand, if in the general baroclinic case the altitude is approximated in a multilevel manner as

$$z(x,y,t,p) = \sum_{i=0}^{N} z_i(x,y,t)\,\psi_i(p), \tag{12.2}$$

then the functions $\psi_i(p)$, $i = 0, 1, \ldots, N$ should be chosen on the basis of a-priori information. For any finite number N, each of the functions $z_i(x, y, t)$ becomes a parameter of a given mathematical model that can, in principle, be expressed in terms of the values of $z(x, y, t, p)$ at the given level of pressure p_i. Hence the question arises as for the optimal a-priori choice of the functions $\psi_i(p)$. Such a choice is multilevel by its nature. It requires an interpolation between given levels on the basis of some qualitative a-priori assumptions. These assumptions have to ensure the well-posedness of the model.

We recall that in short-range climate study, neither external energy generation nor dissipation of energy due to internal processes are taken into account. Let us assume now that h is the smallest scale of motion described by a discretized system of prognostic equations. Firstly, we note that the consideration of a discretized system is natural, at least due to imperfections in the measurement of meteorological fields. In reality, even at very high resolution, the scale h still exceeds the scales of regions of energy dissipation. However, if we neglect the energy dissipation then the energy transferred between scale spectrum finally reaches the scale of the order h and accumulates there without a dissipation. As a result, the non-homogeneities of meteorological fields with scales of the order h may increase in time when $t - t_0$ increases, inducing nonlinear instability. This leads to a continuous correction of the model by a more precise definition of the law of dissipation. From the mathematical point of view, whenever

$$h \to 0 \text{ and/or } t - t_0 \to \infty \tag{12.3}$$

the dependency of h on τ and/or the dependency of $t - t_0$ on τ becomes important.

13. Computational decoupling.

The system (11.1)—(11.4) is a strongly coupled system of mathematical equations. Its solution cannot be obtained by analytical approaches unless substantial simplifications are made. Such simplifications may dramatically influence the validity of the final result. On the other hand, any specific physical parameterization of the model also implies an inevitable mathematical approximation as we explained in Section 12. Due to such an approximation, conservative properties of the original system may only be preserved approximately. The accuracy of approximations of conservation laws is determined by the physical parameterization and the degree of coupling in the original mathematical model. Essentially, any physical parameterization that is "built-in" to a mathematical model splits the model into components. However, in principle, the connections between such components can be restored computationally. The quality of the restoration depends on the number of model components and the quality of the physical parameterization with respect to the real processes and phenomena.

In the Climate System Model developed by NCAR there are four main components, namely atmosphere, ocean, land, and sea-ice. The connection between these components are realized using the Flux Coupler code [5]. This code is constructed under the assumption that conservative properties can be preserved for momentum, heat, and freshwater under message passing. In turn, this assumption inevitably leads to an approximation of the original model. Even if we assume that initial data (initial and boundary conditions) are given with an appropriate precision, additional assumptions for the energy dissipation law at the top of the atmosphere should be made by a-priori arguments. This implies an approximate character not only for the mathematical expression of physical laws, but an approximate character for the physical parameterization of the model as well.

Reasonable a-priori assumptions may be derived on the basis of experimentation and observations. They can provide a tool for the analysis of the adaptive readjustment of meteorological fields. However, a formal expression of such adaptive procedures requires some a-posteriori arguments. Such a-posteriori arguments are usually based on the concept of continuity [17]. Having both a-priori assumptions and a-posteriori arguments we can, in principle, validate the model ensuring its stability. In the general case the validation of mathematical models for complex dynamic systems can only be conducted with incomplete information. As a result, in reality it is practically impossible to achieve 100% reliability of the model. However, it is possible to achieve a balance between the reliability of the model subjected to the physical parameterization and the efficiency of a numerical algorithm for its solution. The procedure for achieving such a balance requires the adaptive error control that is based on a-posteriori information about the computed solution. In turn, the processing of such information requires a-priori information on the exact solution. Under quite general assumptions problems of this type can be formalized mathematically in the form of a hyperbolic type partial differential equation with respect to the control function [35]. From the practical point of view, the well-posedness of the original model depends on a constant of hydrodynamic stability, C^s, that quantifies stability properties of a dual problem with coefficients which depend on exact and computed solutions as well as on the period of time $T = t - t_0$ during which the model is integrated [18]. In this interpretation, the validation of the original mathematical model is eventually determined by the evaluation of the quantity C^s, which for climate system models is closely associated with the relaxation time τ defined by (4.3). We will address the issues of such an evaluation elsewhere. Here we note that the foundation of theory in this direction was laid by the works [17, 7, 18] (see also references therein).

14. Numerical results.

In order to conduct a computational experiment we have used the National Centre for Atmospheric Research model [1]. It can be run in three main modes, namely interactive, batch and message passing. Only the first two were used in our

experiment. The original file contains a C-shell script "setup" that can be used for the configuration of the model (type of dynamics, resolution etc). The "setup" creates a directory with the configuration specific name, for example, in one of our cases it was "cray.t42.spectral.som/". On a SUN station with the SunOS operating system at the CIAM, University of South Australia we compiled the model with the command: "make sunos". This creates the executable file "ccm3bin" in a subdirectory "run/". Boundary datasets were taken from the NCAR WWW domain in the IEEE binary format. In the interactive mode the "setup" generates two standard namelists for initial and restart runs that can be used to tune in the model. The namelist files can be written at the discretion of a user as described in [1], p.27—44. We also note that running the model on SUN SPARC stations an increase of the stack size is often necessary, subject to the resolution used.

When we have to perform the simulation for a longer period it is convenient to run the model in the batch mode. The Fujitsu VPP—300 supercomputer at the Australian Supercomputer Facility was used for such a simulation. The VPP system is a distributed memory machine. Availability on this machine of the vectorizing and parallelizing UXP/V Fortran-90 [56, 57] compiler and other software capabilities makes this computer effective for the high-speed computation required for a CCM3 run. In this computational experiment only one processor was used to run a vectorized version of the CCM3 code in standard and slab ocean versions. A few bugs reported recently through the CCM-Users E-Mail Group were fixed. For a parallelized code the message passing using the PVM facility should be implemented, which can be seen as a future development of this work.

Below we present typical outputs obtained as a result of climate simulation. In this paper we have not attempted to investigate the error of this simulation. As follow from the above discussion, the total error consists of three parts:

- the error of initial data at the start of computer simulation,
- error of the finite set of differential equations in the description of climate, and
- the error of the numerical algorithm that is used.

For such models as the NCAR CSM, the total error obtained from the contribution from all three sources is practically infeasible. Instead, we are currently developing a technique for the evaluation of such an error for a simplified model. The purpose of numerical results presented here is to give a comprehensive graphical interpretation of several physical fields that have been computed and can be used for future analysis.

In the simulation of climate we used the standard input datasets ([1], p.45—50). As the main output, the model generates so-called history files that are in a binary format. They provides the information on a set of temporal samples. Field values at any given moment of time correspond to different latitudes ([1], p.51). Different plotting programs can be used for the interpretation of the results of outputs. We used a modified version of the code developed at the Global Change Research Cen-

ter, Portland State University by Gerhard W. Gross whose help is gratefully acknowledged. This code reads history files and plots them using the GNUPLOT Plotting Program. In Figs. 1 — 5, typical initial distributions are presented for the following fields respectively

- surface geopotential in m^2/s units;
- surface pressure in Pa units;
- zonal wind component in m/s units;
- meridional wind component in m/s units;
- sea-surface temperature field in C units.

In Figs.6 and 7 zonal and meridional components of the wind are presented at vertical level 15. The temperature field is presented in Fig.8 for vertical level 15 in K units. Finally, the water vapour field at the vertical level 15 in Kg_{H_2O}/Kg_{air} units is presented in Fig.9. The complete Master Field List of the NCAR CCM3 model and available options for the output of the model can be found in [1], p. 45—68.

15. Conclusions and future directions.

The coupled simulation for climate system models provides an efficient tool for climate study. Moreover, the concept of coupling in modelling complex dynamic systems reflects one of the most general ways of implementing new effects and new information into mathematical models. However, the refinements of the approach based on coupling procedures for such complex systems as climate may continue indefinitely. The two natural ways to meet the arising challenge were discussed in this paper. First, it is natural to start with a finite set of independent models and try to couple them by informational message passing using certain physical principles such as conservation laws. The NCAR Climate System Model is of this type. We presented several numerical examples obtained on the basis of this approach. For the NCAR CSM the set of independent components consists of four mathematical models for atmosphere, ocean, land surface, and sea ice. Although the conservative properties of the whole system cannot be guaranteed in general, certain key components, such as momentum, heat, freshwater, can be preserved numerically. The quality of such models depend on the dynamics of the error propagation that can be controlled by the Coupler Code. With additional information becoming available, this approach requires an increasingly complex code. In the end, we have to deal with the coupling phenomenon from the very beginning of the process of modelling. On the other hand, the sparse character of available informational datasets for such complex dynamic systems as climate makes the concept of independency for the model components natural, at least at the initial step of the process of modelling.

The dilemma between independence and coupling has led us to the necessity of considering another approach to the modelling of complex dynamic systems. We came to the conclusion that such an approach has to be based not only on a-priori information about the system (that is incomplete in its nature) but also on a-

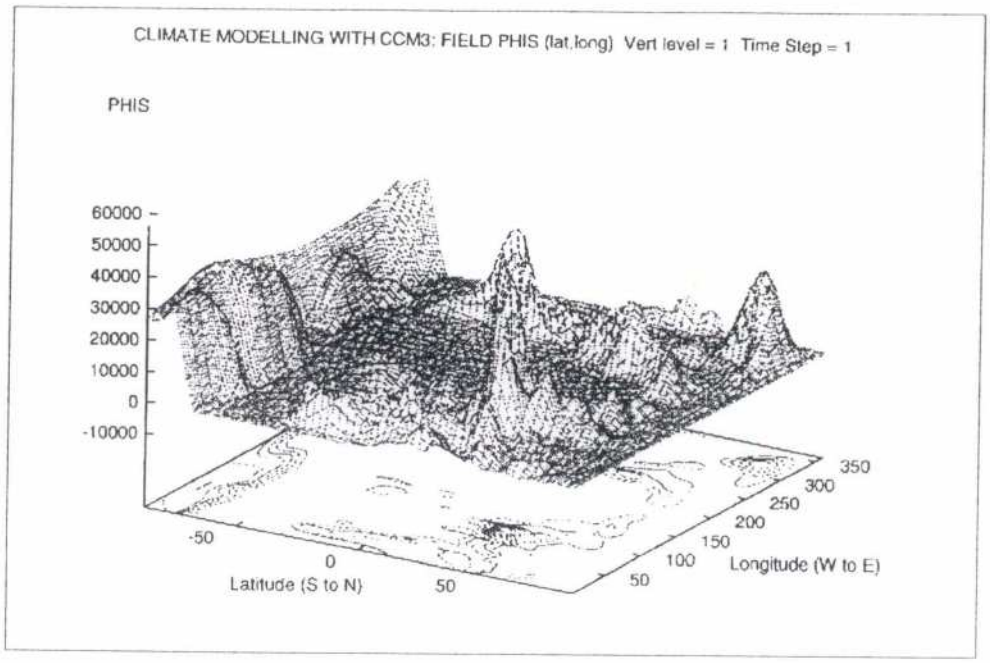

Figure 1. Surface geopotential (PHIS).

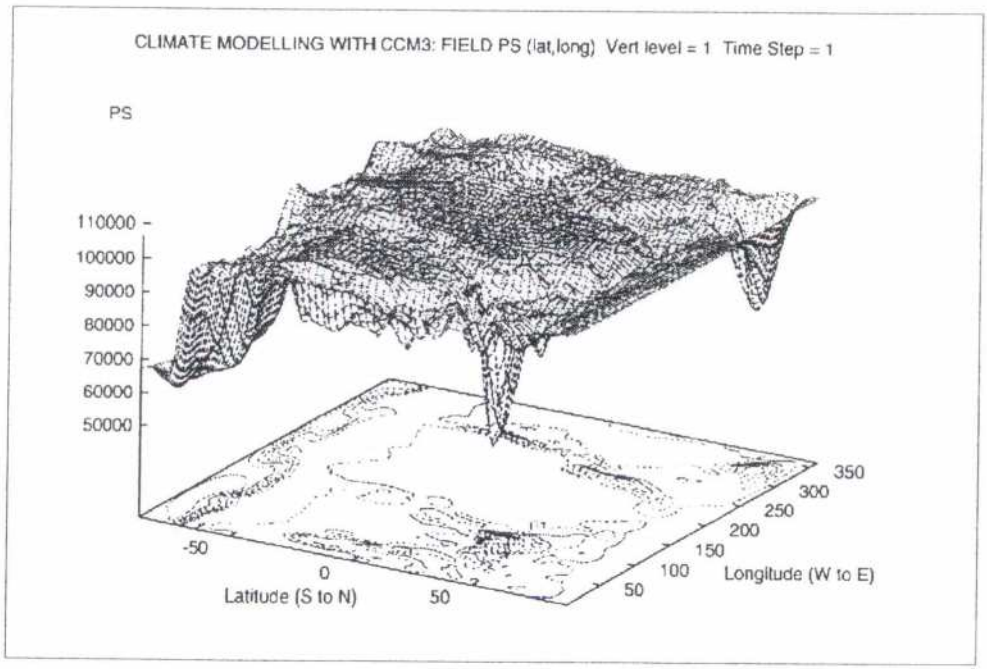

Figure 2. Surface pressure (PS).

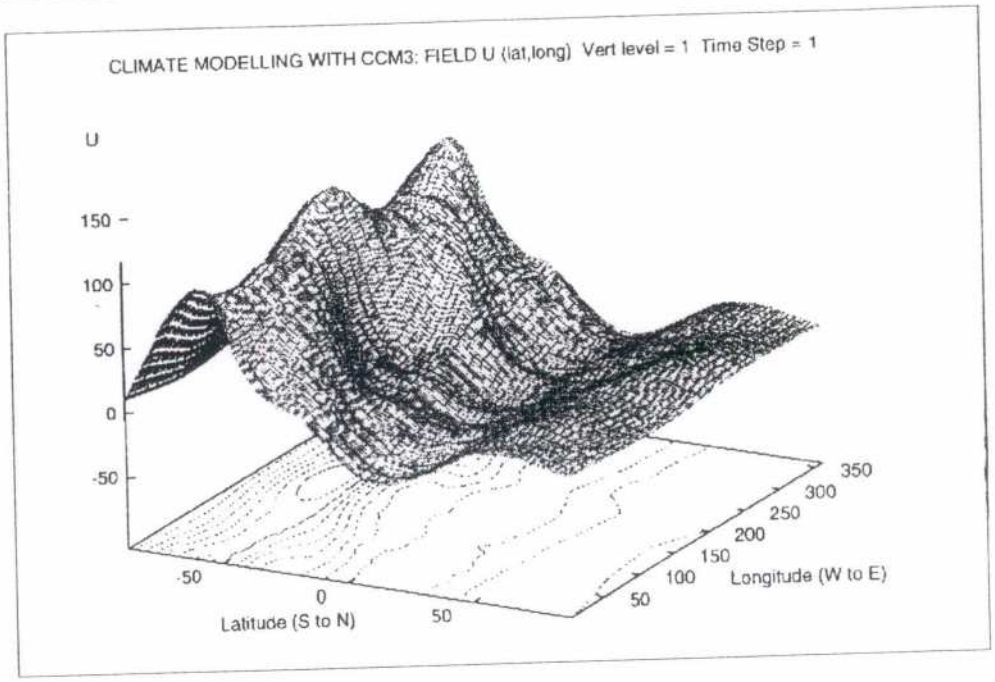

Figure 3. Zonal component of the wind (U, vertical level 1).

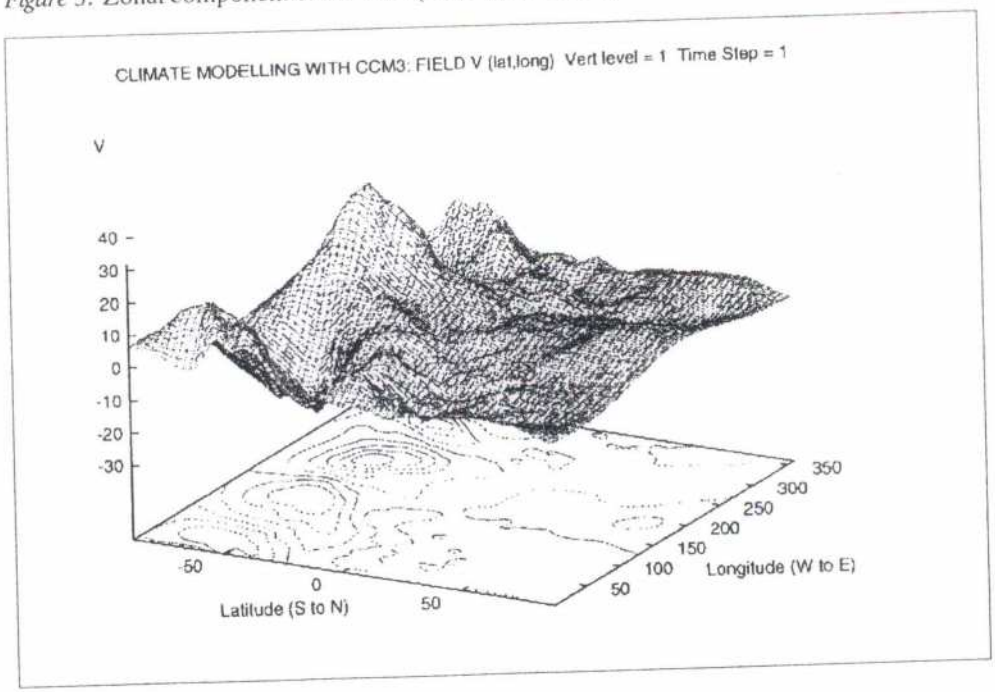

Figure 4. Meridional component of the wind (V, vertical level 1).

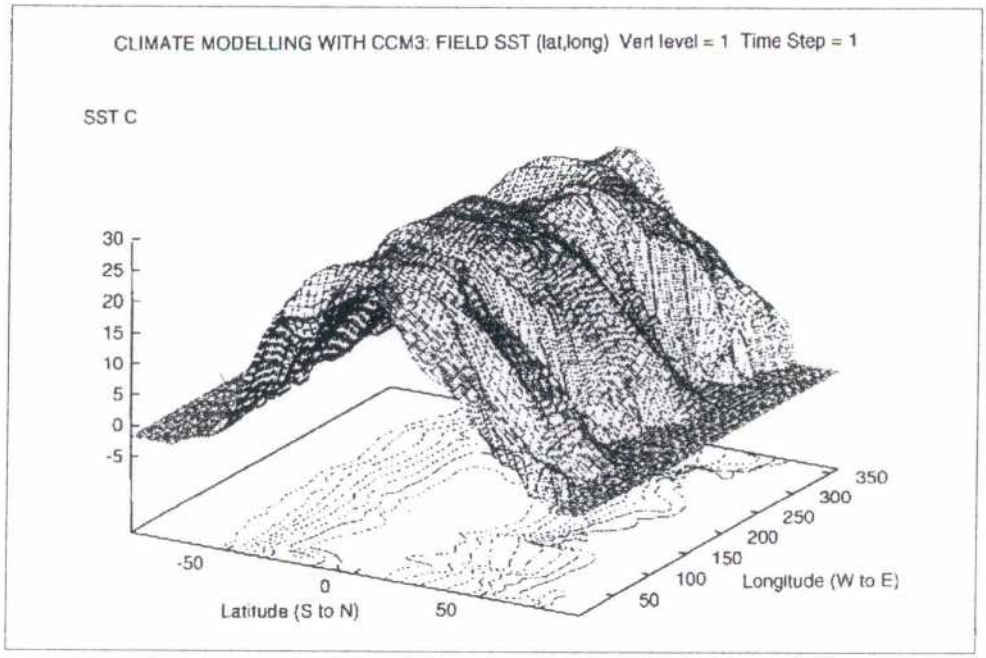

Figure 5. Sea-surface temperature distribution (SST).

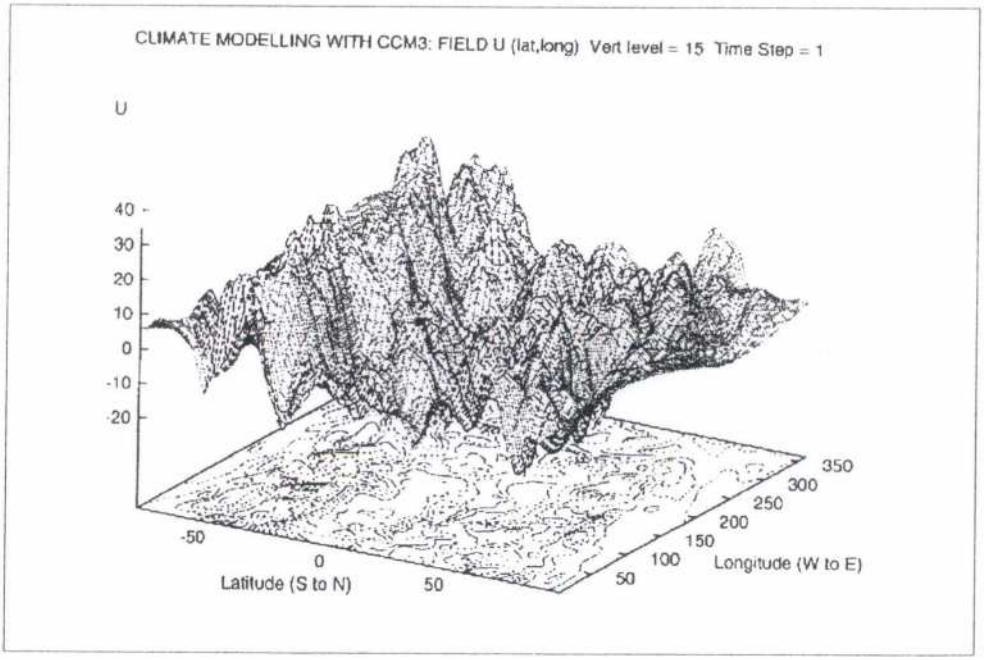

Figure 6. Zonal componen of the wind (U, vertical level 15).

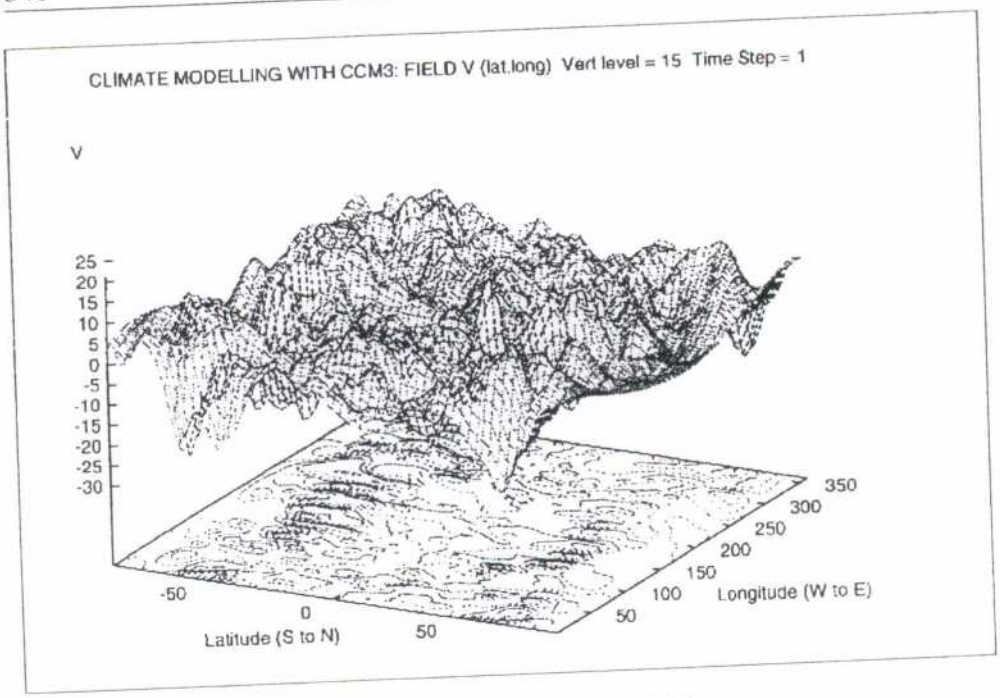

Figure 7. Meridional component of the wind (V, vertical level 15).

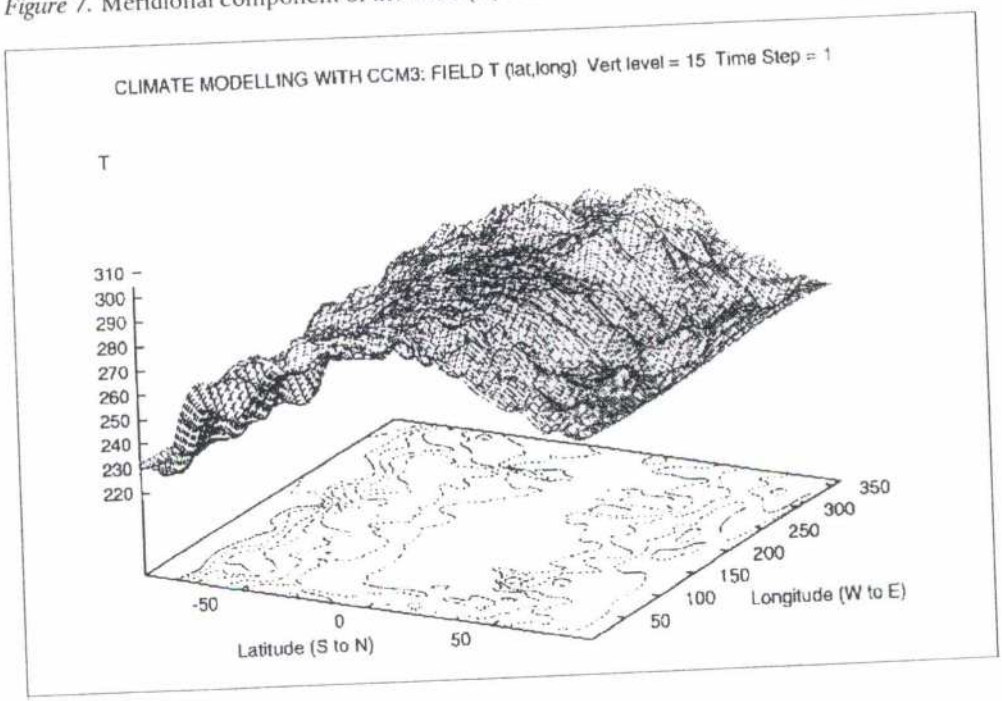

Figure 8. Temperature field at vertical level 15.

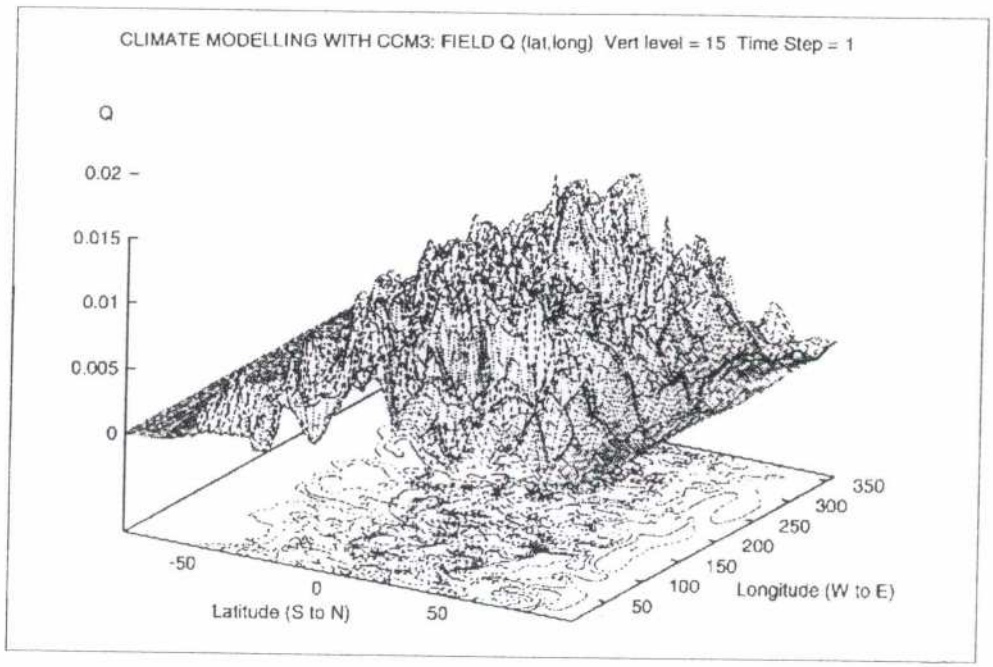

Figure 9. Water vapour at vertical level 15.

posteriori information. This allows us to construct numerical algorithms that can be analyzed in the traditional manner of using a-priori information, but the procedure of construction has to be based on a-posteriori information. In this case the choice of the norm for the error control is model-specific, being influenced by the physical parameterization of the mathematical model. For any given physical parameterization, conservation laws may be implemented only approximately into mathematical models of complex dynamic systems. As a result, the standard energy norms may not provide an appropriate choice for the error control in mathematical models of such systems. This idea is the basis for the future development of the presented work. Under a fixed degree of coupling and a given physical parameterization we need a scale of a-priori and a-posteriori estimates in a spectrum of norms to ensure the stability of the model. Numerical algorithms based on such estimates allow an adaptive error control within the chosen spectrum. In turn, this allows the construction of adaptive computational codes that can be effectively used in the study of complex dynamic systems with many transient states.

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REFERENCES

1. T. L. Acker, et al, User's Guide to NCAR CCM3, NCAR TN—421, May, 1996.
2. G. I. Barenblatt and A. J. Chorin, Scaling laws and vanishing viscosity limits in turbulence theory, Department of Mathematics, University of California, Berkeley, 1996.
3. G. I. Barenblatt, A. J. Chorin, and V. M. Prostokishin, Scaling laws for Fully Developed Turbulent Flow in Pipes: Discussion of Experimental Data, Department of Mathematics, University of California, Berkeley, 1996.
4. B. P. Briegleb, Delta-Eddington Approximation for Solar Radiation in the NCAR Community Climate Model, Journal of Geophysical Research, Vol. 97, No. D7, 1992, pp. 7603—7612.
5. P. O. Bryan, et al., The NCAR CSM Flux Coupler, NCAR TN, May, 1996.
6. P. Chang, and S. G. Philander, A coupled ocean-atmosphere instability of relevance to the seasonal cycle, Journal of the Atmospheric Sciences, 51, No. 24, 1994, pp. 3627—3648.
7. K. Eriksson and C. Johnson, Adaptive finite element methods for parabolic problems V: long-time integration, SIAM J. Numer. Anal., 32, No. 6, 1995, pp. 1750—1763.
8. B. F. Farrell, Small Error Dynamics and the Predictability of Atmospheric Flows, Journal of the Atmospheric Sciences, 47, No. 20, 1990, pp. 2409—2416.
9. W. H. Fleming and H. M. Soner, Controlled Markov Processes and Viscosity Solutions, Springer-Verlag, 1993.
10. I. M. Gelfand and N. Ya. Vilenkin, Generalized Functions. Applications to Harmonic Analysis, Academic Press, Vol. 4, 1964, Chapter 4, Section 5.
11. S. L. Hess, Introduction to Theoretical Meteorology, Holt, Rinehart & Winston, 1959, New York.
12. J. L. Holloway, Jr. and S. Manabe, Simulation of Climate by a Global General Circulation Model, Monthly Weather Review, Vol. 99, No. 5, 1971, pp. 335—370.
13. A. A. M. Holtslag and B. A. Boville, Local Versus Nonlocal Boundary-Layer Diffusion in a Global Climate Model, Journal of Climate, Vol. 6, 1993, pp. 1825—1842.
14. J. Honerkamp, Stochastic Dynamical Systems. Concepts, Numerical Methods, Data Analysis. New York: VCH, 1994.
15. E. Hopf, Statistical hydrodynamics and functional calculus, J. Rat. Mech. Anal., 1, 1952, pp. 87—142.
16. F. F. Jin and J. D. Neelin, Modes of interannual tropical ocean-atmosphere interaction — a unified view, Journal of Atmospheric Sciences, 50, No. 21, 1993, pp. 3477—3503.
17. J. R. Johnson, A new paradigm for adaptive finite element method, in The Mathematics of Finite Elements and Applications, Ed. by J. R. Whiteman, 1994, pp. 105—120.
18. C. Johnson, R. Rannacher and M. Boman, Numerics and hydrodynamic stability: toward error control in computational fluid dynamics, SIAM J. Numer. Anal., 32, No. 6, 1995, pp. 158— 1079.

19. A. Kasahara and W. Washigton, NCAR Global General Circulation Model of the Atmosphere, Monthly Weather Review, Vol. 95, No. 7, 1967, pp. 389—402.
20. A. Kasahara, Various Vertical Coordinate Systems Used for Numerical Weather Prediction, Monthly Weather Review, Vol. 102, 1974, pp. 509—522.
21. J. T. Kiehl et al., Description of the NCAR Community Climate Model (COM3), NCAR TN —420,1996.
22. A. N. Kolmogorov, Local structure of turbulence in an incompressible fluid at a very high Reynolds number, Dokl. Akad. Nauk. SSSR, 30, 1941, pp. 299—302.
23. H. O. Kreiss, Problems with different time scales, in Acta Numerica, 1992, pp. 101—139.
24. Y. Kurihava, Numerical Integration of the Primitive Equations on a Spherical Grid, Monthly Weather Review, Vol. 93, No. 7, 1965, pp. 399—415.
25. P. D. Lax, Hyperbolic systems of conservation laws and the mathematical theory of shock waves, SIAM Publications, Philadelphia, 1972.
26. P. D. Lax, The zero dispersion limit, a deterministic analog of turbulence, Comm. Pure Appl. Math., 44, 1991, pp. 1047—1056.
27. T. D. Lee, On some statistical properties of hydrodynamic and hydromagnetic fields, Quarterly AppL Math., 1952, pp. 69—72.
28. S.-J.Lin et al., A class of the van Leer-type transport schemes and its application to the moisture transport in a General Circulation Model, Monthly Weather Review, 122, 1994, pp. 1574—1592.
29. C.-C. Ma et al., Sensitivity of a Coupled Ocean-Atmosphere Model to Physical Parameterizations, Journal of Climate, Vol. 7, 1994, pp. 1883—1896.
30. S. Manabe, J. Smagorinsky and R. Strickler, Simulated Climatology of a General Circulation Model with a Hydrological Cycle, Monthly Weather Review, Vol. 95, No. 7, 1967, pp. 389—402.
31. S. Manabe, Climate and the Ocean Circulation, Monthly Weather Review, Vol. 97, No. II, 1969, pp. 739—774.
32. G. I. Marchuk, Splitting and Alternating Direction Methods, in Handbook of Numerical Analysis, Vol. I, Ed. by P. G. Ciarlet and J. L. Lions, 1990, Elsevier Science Publishers, pp. 199— 462.
33. J. P. McCreary,Jr. and D. L. T. Anderson, An Overview of Coupled Ocean-Atmosphere Models of EI-Nino and the Southern Oscillation, Journal of Geophysical Research, 96, No. 28, 1991, pp. 3125—3150.
34. V. N. Melnik, Semi-implicit finite-difference schemes with flow correction for quasihydrodynamic models, Engineering Simulation, 1995, pp. 856—865.
35. V. N. Melnik, Nonconservation law equation in mathematical modelling: aspects of approximation, Proc. of the International Conf. AEMC'96, Sydney, Eds. W. Y. D. Yuen, P. BroadBridge and J. M. Steiner, 1996, pp. 423—430.
36. V. N. Melnik, Optimal probabilistic trajectories of deterministic finite-state machines, School of Mathematics, University of South Australia, TR 15, 1996, pp. 1—25. (to appear in Discrete Dynamics in Nature and Society under the title "Dynamic System Evolution and Markov Chain Approximation").
37. A. S. Monin, Weather Forecasting as a Problem in Physics, The MIT Press, 1972.
38. I. Muller and T. Ruggeri, Extended Thermodynamics, Springer-Verlag, 1993.
39. A. M. Obukhov, Spectral energy distribution in turbulent flow, Dokl. Akad. Nauk SSSR, 32, 1941, pp. 22—24.
40. V. A. Peshkov, Second sound in helium II, Journal of Physics, 3, 1944, 381.

41. N. A. Phillips, A Coordinate System Having Some Special Advantages for Numerical Forecasting, Journal of Meteorology, Vol. 14, 1957, pp. 184—185.
42. L. Prandtl, Bericht uber Untersuchungen zur ausgebildeten Turbulenz, Zeitschr. angew. Math. Mech., 5, 1925, 136—139.
43. L. Prandtl, Zur turbulenten Stroemung in Rohren und laengs Platten, Ergeb. Aerodyn. Versuch., Series 4, 1932, Goettingen.
44. Yu. V. Prokhorov, Convergence of random processes and limit theorems of probability theory, Theory Probab. and AppL, 1, 1956, pp. 157—214.
45. D. A. Randall et al., Analysis of snow feedback in 14 general circulation models, Journal of Geophysical Research, 99, No. DIG, 1994, pp. 20757—20771.
46. P. J. Rasch and D. L. Williamson, The Sensitivity of a General Circulation Model Climate to the Moisture Transport Formulation, Journal of Geophysical Research, 96, No. D7, 1991, pp. 13123—13137.
47. P. J. Rasch et al., A three-dimensional transport model for the middle atmosphere, Journal of Geophysical Research, 99, No. Dl, 1994, pp. 999—1017.
48. P. L. Read, Applications of chaos to meteorology and climate, in The Nature of Chaos, Ed. by T. Mullin, Clarendon Press, Oxford, 1994, pp. 222—260.
49. C. A. Reynolds, P. J. Webster and E. Kalnay, Random error Growth in NMC's Global Forecasts, Monthly Weather Review, 122, 1994, pp. 1281—1305.
50. L. F. Richardson, Weather Prediction by numerical Process Cambridge Univ. Press, London, 1922.
51. V. Sazonov, A note on characteristic functionals, Theory Probab. and Appl., 3, 1958, pp. 188—192.
52. J. Smagorinsky, General Circulation Experiments with the Primitive Equations, Monthly Weather Review, Vol. 91, No. 3, 1963, pp. 99—164.
53. M. Shashkov, Conservative Finite-Difference Methods on General Grid, CRC Press, 1996.
54. M. Struwe, Variational Methods. Applications to Nonlinear PDEs and Hamiltonian Systems, Springer-Verlag, 1990.
55. G. J. Tripoli, A nonhydrostatic mesoscale model designed to simulate scale interaction, Monthly Weather Review, 120, 1992, pp. 1342—1359.
56. UXP/V Fortran-90/VP. User's Guide (VIO), J2U5—0050-OIEN, Fujitsu, 1995.
57. UXP/V Fortran-90 Messages (VIO), J2U5-0060-OIEN, Fujitsu, 1995.
58. P. Vernotte, La veritable equation de la chaleur, Comtes Rendus Hebd. Seances Acad. Sci., 247(23), 1958, pp. 2103—2105.
59. T. Von Karman, Mechanische Aehniichkeit und Turbulenz, Nach. Ges. Wiss. Goettingen Math.-Phys. Klasse, 1932, 58—76.
60. Y. Wakata, and E. S. Sarachik, Nonlinear Effects in Coupled Atmosphere-Ocean Basin Modes, Journal of Atmospheric Sciences, 51, No. 6, 1994, pp. 909—920.
61. D. L. Williamson and J. G. Olson, Climate simulation with a semi-Lagrangian version of the NCAR Community Climate Model, Monthly Weather Review, 122, 1994, pp. 1595—1610.
62. G. Wu, and N. C. Lau, A GCM simulation of the relationship between tropical-storm formulation and ENSO, Monthly Weather Review, 120, 1992, pp. 958—977.