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A comparative analysis of modal motions for the gyroscopic and non-gyroscopic two degree-of-freedom conservative systems

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ABSTRACT

The synchronous in-unison motions in vibrational mechanics and the non-synchronous out-of-unison motions are the most frequently found periodic motions in every fields of science and everywhere in the universe. In contrast to the in-unison normal modes, the out-of-unison complex modes feature a $\pi/2$ phase difference. By the complex mode analysis we classify the out-of-unison planar motion into two types, gyroscopic motions and elliptic motions. It is found that the gyroscopic and elliptic motions have different characteristics for a two degree-of-freedom (2DOF) system. The gyroscopic motion involves two distinct frequencies with, respectively, two corresponding complex modes. However, the elliptic motion the nonlinear non-gyroscopic 2DOF system with repeated frequencies involves only single frequency with corresponding two complex modes. The study of the differences and similarities of the gyroscopic and elliptic modes sheds new light on the in-depth mechanism of the planar motions in the universe and the man-made engineering systems.

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1. Introduction

Nonlinear normal modes of the conservative oscillators are defined as synchronous periodic motions by Rosenberg [1,2], during which all coordinates of the system vibrate in-unison, reaching their maximum and minimum values at the same instant of time. The free and forced dynamic responses for arbitrary sets of initial conditions can be derived analytically by the superposition principle in the normal modes framework for the linear oscillating systems.

One typical two-of-freedom (2DOF) system is presented in Fig. 1. The system can be considered as linear if the geometric nonlinear effect is assumed small and therefore negligible for the case of small amplitude vibrations. The in-line straight motions of the 2DOF system are typically in-unison or normal mode motions. The out-of-line motions around the original

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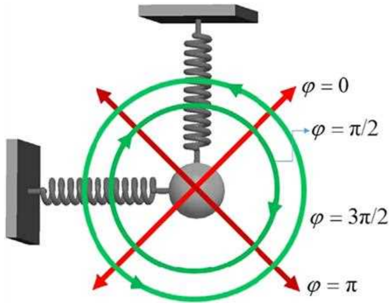

Fig. 1. The orbits of normal modes (NM) and elliptic modes (EM). NM (in red lines): $\varphi=0$, in-phase linear motion; $\varphi=\pi$, anti-phase linear motion. EM (in green circles): $\varphi=\pi/2$, clockwise elliptic motion; $\varphi=3\pi/2$, anti-clockwise elliptic motion. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

point are usually called out-of-unison or, particularly, elliptic mode motions. Unlike in-unison motions, the out-of-unison vibrations involve complex modes. In the 2DOF system, the out-of-unison motions behave in such a way that while one coordinate reaches the extremum, the other passes through zero, so that they are $\pi/2$ out-of-phase. The trajectories of the in-unison motions are presented by the red straight-lines in Fig. 1, and the out-of-unison motions are presented by the circular orbits.

For the 2DOF system with coordinates on perpendicular directions, it is natural to define the in-line normal motions and out-of-line elliptic motions. For the case all the coordinates are in parallel, we still can locate the normal and elliptic motions by the phase lag. Krack et al. [3,4] and Blanchard et al. [5] studied such mode complexity by the phenomena of standing waves and traveling waves, which demonstrate the normal and elliptic motions, respectively.

We can observe 2DOF planar periodic motions, or almost periodic motions in every fields of science and everywhere in the universe, from the macroscopic revolution of the Earth around the Sun to the microscopic rotations of electrons around the nuclei [6]. Elliptic (circular as a particular case) motion is one of the most common out-of-unison periodic motions. Yang et al. [7] investigated the elliptic motions of a rotor suspending in active magnetic bearings, which is a typical non-gyroscopic nonlinear 2DOF system with repeated frequencies. Another type of out-of-unison periodic motions are gyroscopic motions, which demonstrate special features compared to the elliptic motions. The two-body problem in celestial dynamics and the rotations of electrons in quantum mechanics belong to the type of non-gyroscopic elliptic motions. At the same time, the restricted three-body problem[8] and the orbit dynamics around the libration points, can be modeled as gyroscopic motions. Other typical gyroscopic systems include spinning tops, axially moving continuum [9,10], pipes conveying fluid [11,12], and whirling motions of the rotating elastic shafts [13,14], which give rise to Coriolis force.

In recent years, there have been a number of investigations related to the concept of nonlinear normal modes [15–18], which shed a new light on the modal analysis of nonlinear out-of-unison systems. This new developments are largely due to earlier works by Shaw and Pierre [19,20] who introduced the invariant manifold method to derive the nonlinear normal modes which are not confined to the concept of vibration in-unison modes of Rosenberg.

Although the elliptic and gyroscopic motions belong to the class of the so-called out-of-unison motions or non-synchronous motions [18], they have different features. In this study, we classify the periodic planar motions in the universe and man-made engineering systems, into three types, namely, traditional in-unison normal motions, out-of-unison elliptic motions and gyroscopic motions. We construct the out-of-unison motions modeled by simple nonlinear 2DOF oscillators with repeated frequencies and discuss the differences and similarities of the gyroscopic and elliptic motions based on the complex mode analysis, which has been overlooked by the physicists.

2. The modal motions of 2DOF gyroscopic systems

The governing ordinary differential equations for the linear conservative gyroscopic systems can be written as

$$\mathbf{M}\ddot{\overline{\mathbf{x}}}+\mathbf{G}\dot{\overline{\mathbf{x}}}+\mathbf{K}\overline{\mathbf{x}}=\mathbf{0}, \tag{1}$$

where **M**, **G**, and **K** are, respectively, mass, gyroscopic, and stiffness matrices. If $\mathbf{G}=0$, Eq. (1) recovers the governing equations of the non-gyroscopic conservative system.

Next, we will study the modal motions and general motions for the linear and non-gyroscopic 2DOF systems, respectively.

2.1. The motions of the linear gyroscopic 2DOF system

The coupled stiffness part can be decoupled as the system (1) being transferred from the physical coordinate frame to the primary coordinate frame by introducing $\mathbf{x}=\mathbf{P}\overline{\mathbf{x}}$, where **P** is the matrix composed by the eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$. The stiffness

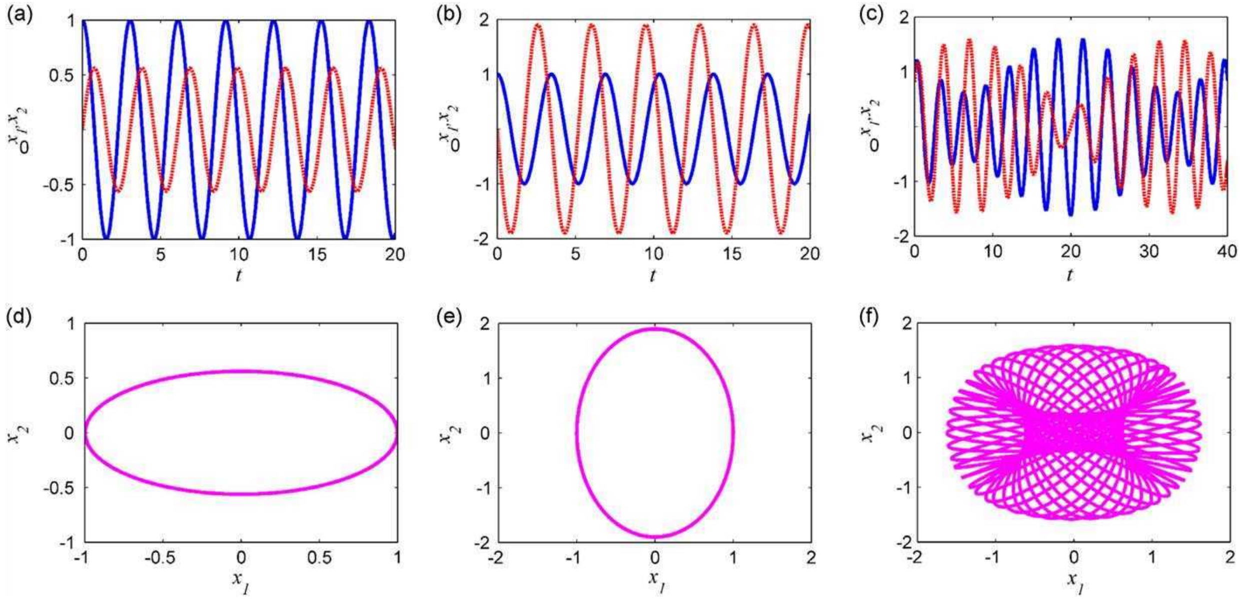

Fig. 2. The time histories and phase trajectories of the gyroscopic system ($k_1=4$, $k_2=3.5$, $g=0.2$) (a), (b), and (c) are the time histories of elliptic mode I, II, and transient motion, respectively; (d), (e), and (f) are the phase trajectories of elliptic mode I, II, and transient motion, respectively.

decoupled gyroscopic 2DOF system can be written as

$$\begin{aligned} \ddot{x}_1 + g\dot{x}_2 + k_1 x_1 = 0, \\ \ddot{x}_2 - g\dot{x}_1 + k_2 x_2 = 0, \end{aligned} \tag{2}$$

where g denotes the measure of the gyroscopic effect, and k_1 and k_2 are positive stiffness parameters. The two natural frequencies of system (2) are

$$\omega_{1,2} = \sqrt{\tfrac{1}{2}\left(k_1 + k_2 + g^2\right) \mp \tfrac{1}{2}\sqrt{\left(k_1 + k_2 + g^2\right)^2 - 4k_1 k_2}} \tag{3}$$

and the corresponding two amplitudes ratios of the complex modes are

$$A_1^{(1)} : A_2^{(1)} = 1 : \frac{ig\omega_1}{k_2 - \omega_1^2}, \quad A_1^{(2)} : A_2^{2} = 1 : \frac{ig\omega_2}{k_2 - \omega_2^2}. \tag{4}$$

where the superscript in parenthesis mean the corresponding mode order. Since the ratio between the two oscillators are pure imaginary as demonstrated by (4), the two coupled oscillators have a $\pi/2$ phase difference for such gyroscopic 2DOF system. During the modal motions, the first oscillator reaches its maximum value, and the second oscillator goes through its equilibrium point with maximum velocity. It appears that the displacement of one oscillator is proportional to the velocity of the other oscillator. Because of the $\pi/2$ phase difference, the system behaves like the second oscillator following the first one with a quarter period gap, or vice versa. It is concluded that the two frequencies of the system are distinct if the gyroscopic parameter g is not zero. Even for the identical stiffness, $k_1=k_2=k$, we still have two distinct frequencies.

Based on the imaginary amplitude ratio (4), during modal motions, the elliptic modes, as presented in Fig. 2(a) and (d) and (b) and (e), can be plotted based on the initial conditions

$$\begin{pmatrix} x_1 & \dot{x}_1 & x_2 & \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dfrac{-g\omega_n^2}{k_2 - \omega_n^2} \end{pmatrix}, \; n = 1,\ 2. \tag{5}$$

The transient motions are those not coinciding with the modal motions, which, in the linear case, can be regarded as the superposition of the two modal motions. The transient motions are usually quasi-periodic if the two order natural frequencies are not commensurable, otherwise, they are multiple-periodic motions. The contours of the transient motions are confined in an elliptic area as shown in the Fig. 2(c) and (f).

In the time histories presented in Fig. 2, the phase differences $(\varphi_2 - \varphi_1)$ of $\pi/2$ and $3\pi/2$ can be clearly located for the elliptic modal motions, respectively. However, the phase differences, and amplitude ratio are time varying for the transient motions although the total energy remains constant.

It should be noted that if the stiffness terms are coupled in the gyroscopic system (1), the corresponding elliptic modes and the transient motion contour will be shown in a slanted manner. The phase difference of the two oscillators for the elliptic modes is not exactly $\pi/2$.

2.2. The motions of the nonlinear gyroscopic 2DOF system

The conservative weakly nonlinear gyroscopic 2DOF system is governed by the following equations

$$\begin{aligned}\ddot{x}_1+g\dot{x}_2+k_1x_1&=\varepsilon\sum_{n=2}^{\infty}\sum_{k=0}^{n}\alpha_{n,k}x_1^kx_2^{(n-k)},\\ \ddot{x}_2-g\dot{x}_1+k_2x_2&=\varepsilon\sum_{n=2}^{\infty}\sum_{k=0}^{n}\beta_{n,k}x_1^{(n-k)}x_2^k.\end{aligned}\tag{6}$$

where ε is a bookkeeping device to indicate the small nonlinear terms.

By the invariant manifold method proposed by Shaw and Pierre [19,21], the displacement and velocity pair of the second oscillator is related to that of the first oscillator. For the linear case, the amplitude (velocity) of one direction is linearly related to the velocity (amplitude) of the other direction. Hence, during the modal motions of the weakly nonlinear gyroscopic system, we can assume the following approximate nonlinear equations

$$\begin{cases}x_1=u,\\ \dot{x}_1=v,\end{cases}$$
$$\begin{cases}x_2=a_1u+a_2v+a_3u^2+a_4uv+a_5v^2+a_6u^3+a_7u^2v+a_8uv^2+a_9v^3\cdots,\\ \dot{x}_2=b_1u+b_2v+b_3u^2+b_4uv+b_5v^2+b_6u^3+b_7u^2v+b_8uv^2+b_9v^3+\cdots.\end{cases}\tag{7}$$

The invariant manifold based on Eq. (7) constructs a relationship between the first and second oscillators during modal motions. To solve for the nonlinear modes of such system, the only thing we should do is to determine the coefficients in (7). There are two relations which need to be considered in order to derive these coefficients. The first condition, may be called a compatibility relation, is obtained by equaling the derivatives of the first and third equations to the second and the fourth ones, respectively, in (7). The second relation, of course, should consider the governing Eq. (6). Substituting the derivatives of the second and the fourth equations of (7) back into (6), and replacing the variables of the results by u and v yield the second relation. Expand all the equations of the two relations into polynomials of u,v, and letting the gathered coefficients of each term be zero, we can determine the unknown coefficients a_1, a_2, …, and b_1, b_2, ….

Now we propose the following example to illustrate modal motions of the nonlinear gyroscopic system:

$$\begin{aligned}\ddot{x}_1+g\dot{x}_2+k_1x_1&=k_{11}x_1^3+k_{12}x_1x_2^2,\\ \ddot{x}_2-g\dot{x}_1+k_2x_2&=k_{21}x_2x_1^2+k_{22}x_2^3.\end{aligned}\tag{8}$$

The 3-order approximate invariant manifold during the modal motions can be assumed as

$$\begin{aligned}x_1&=u,\quad \dot{x}_1=v,\\ x_2&=a_1u+a_2v+a_3u^3+a_4u^2v+a_5uv^2+a_6v^3,\\ \dot{x}_2&=b_1u+b_2v+b_3u^3+b_4u^2v+b_5uv^2+b_6v^3,\end{aligned}\tag{9}$$

where the square polynomial terms have been neglected since it has been verified that all the coefficients before the square terms are zero due to the fact that this example involves only cubic nonlinear terms.

Applying the above procedure, we can obtain analytically the two sets of solutions of the coefficients corresponding to the two modes as

$$\begin{gathered}a_1=a_3=a_5=b_2=b_4=b_6=0\\ a_2=\frac{-g^2+k_2-k_1\mp\sqrt{(g^2+k_1-k_2)^2+4k_2g^2}}{2gk_2}\\ b_1=\frac{k_1\left(-g^2+k_2-k_1\mp\sqrt{(g^2+k_1-k_2)^2+4k_2g^2}\right)}{\left(-g^2-k_2-k_1-\sqrt{(g^2+k_1-k_2)^2+4k_2g^2}\right)g}\\ a_4=\frac{s_4s_8+s_5}{2},\ \ a_6=\frac{-w_{12}+s_3}{k_2},\ \ b_3=-\frac{s_2}{s_1}\frac{s_4w_{12}+s_5}{2},\ \ b_5=-\frac{1}{s_6}\left(\frac{s_2}{s_1}-k_{21}\right)\end{gathered}\tag{10}$$

where

$$\begin{gathered}s_1=1-a_2g, s_2=2k_1-gb_1, s_3=-k_{22}(a_2)^3, s_4=s_1-\frac{3}{k_2}(k_1-gb_1)\\ s_5=3\frac{s_3}{k_2}(k_1-gb_1)-k_{21}a_2, s_6=\frac{s_4}{2}\left(\frac{s_2}{s_1}-k_2\right)+2(k_1-gb_1),\\ s_7=\frac{s_5}{2}\left(\frac{s_2}{s_1}-k_{21}\right)-k_{21}a_2, s_8=-\frac{s_7}{s_6}.\end{gathered}\tag{11}$$

Based on the above coefficients, the relationship between the two directions becomes

$$x_2 = a_2 v + a_4 u^2 v + a_6 v^3,$$
$$\dot{x}_2 = b_1 u + b_3 u^3 + b_5 u v^2. \tag{12}$$

These relations presented in (12) define the so-called nonlinear normal modes. Actually, they are not 'normal' since the mode functions are not orthogonal in the complex domain. Indeed, substituting Eq. (12) back into (8) yields

$$\ddot{u} + \omega_n{}^2 u = (gb_3 - k_{11})_n u^3 + (gb_5 - a_2)_n u\dot{u}^2, \; n = 1, \; 2 \tag{13}$$

where n denotes the first or the second mode. The corresponding two linear natural frequencies are

$$\omega_{1,2} = \sqrt{\frac{1}{2}\left(k_1 + k_2 + g^2\right) \mp \frac{1}{2}\sqrt{\left(k_1 + k_2 + g^2\right)^2 - 4k_1 k_2}} \tag{14}$$

Hence, the 2DOF system is decoupled by the invariant manifold method. The nonlinear 1DOF Eq. (13) can describe the nonlinear 2DOF system approximately by the nonlinear transformation. The other dimension of the manifold is related to the one governed by (13) with a $\pi/2$ phase difference for the primary harmonics.

It can be concluded that the nonlinear gyroscopic 2DOF system yields only elliptic modal motions if the higher order harmonics are excluded, which is the case only for a small proportion of the current weakly nonlinear systems.

3. The modal motions of non-gyroscopic 2DOF systems with distinct frequencies

A general linear non-gyroscopic system, in which the stiffness matrix is coupled, can be written as

$$\ddot{\bar{x}}_1 + k_{11}\bar{x}_1 + k_{12}\bar{x}_2 = 0,$$
$$\ddot{\bar{x}}_2 + k_{21}\bar{x}_1 + k_{22}\bar{x}_2 = 0, \tag{15}$$

which yields the normal modes

$$A_1^{(1)} : A_2^{(1)} = 1 : \frac{k_{22} - k_{11} + \sqrt{(k_{22} - k_{11})^2 + 4k_{12}k_{21}}}{2k_{12}},$$
$$A_1^{(2)} : A_2^{(2)} = 1 : \frac{k_{22} - k_{11} - \sqrt{(k_{22} - k_{11})^2 + 4k_{12}k_{21}}}{2k_{12}} \tag{16}$$

The normal modes present real ratios if the stiffness matrix is symmetric and positive definite.

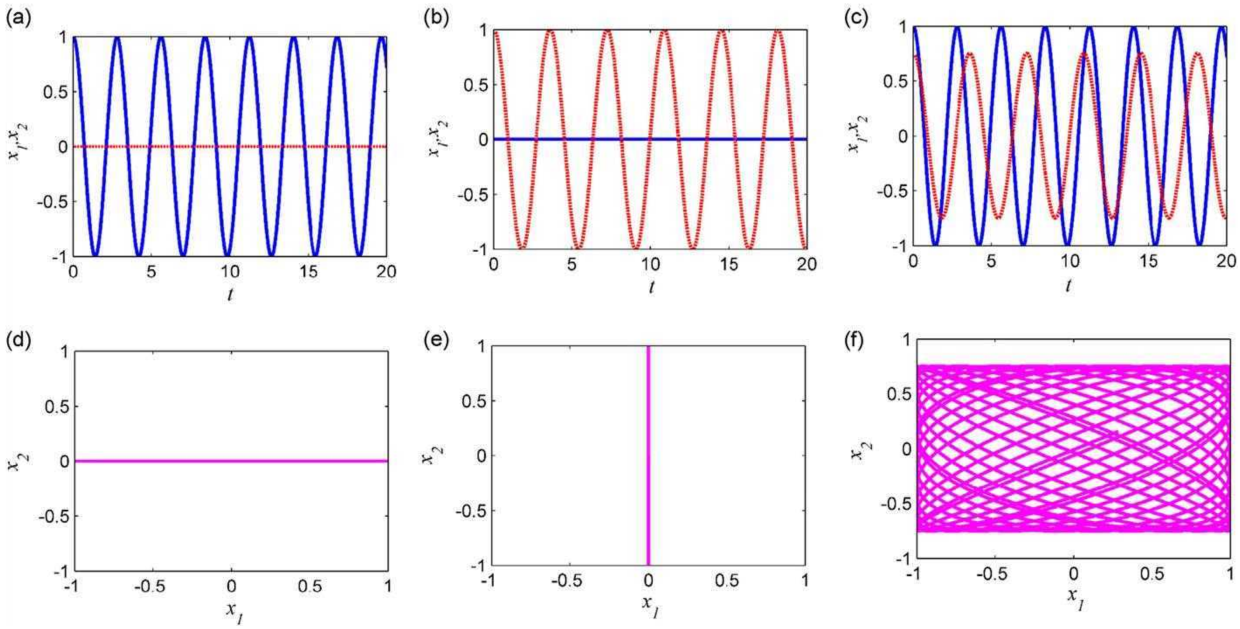

Fig. 3. The time histories and phase trajectories of the non-gyroscopic system ($k_1=5$, $k_2=3$) (a), (b), and (c) are the time histories of normal mode I, II, and transient motion, respectively; (d), (e), and (f) are the phase trajectories of normal mode I, II, and transient motion, respectively.

3.1. The motions of the linear non-gyroscopic 2DOF system

The coupled linear system (15) can be decoupled by the eigenvectors of (16) as

$$\begin{aligned}\ddot{x}_1 + k_1 x_1 &= 0\\ \ddot{x}_2 + k_2 x_2 &= 0\end{aligned} \tag{17}$$

For the case of $k_1 \neq k_2$, the time histories and phase trajectories are also plotted in Fig. 3 for the uncoupled 2DOF system with distinct frequencies. As it will be demonstrated later, the nonlinear generalization of such 2DOF system with repeated frequencies gives rise to a rich dynamics.

It is easy to find that the modal motions are actually single oscillator motions as presented in Fig. 3(a) and (d) and (b) and (e). The transient motions being superposed by the two modal motions are illustrated in Fig. 3(c) and (f). The contour of the quasi-periodic phase trajectory is confined in the rectangular area determined by the initial conditions. If we study the coupled system (15), the normal modes and the transient motion contour are slanted due to the linear transformation (16). Comparing to the transient motions of the gyroscopic systems, we observe that the amplitude of non-gyroscopic transient motions is NOT time-varying. Only the phase difference is changing with respect to time since the two frequencies are not commensurable.

3.2. The motions of the nonlinear non-gyroscopic 2DOF system

The conservative non-gyroscopic nonlinear 2DOF system can be modeled by the following system:

$$\begin{aligned}\ddot{x}_1 + k_1 x_1 &= \varepsilon \sum_{n=2}^{\infty} \sum_{k=0}^{n} \alpha_{n,k} x_1^k x_2^{(n-k)},\\ \ddot{x}_2 + k_2 x_2 &= \varepsilon \sum_{n=2}^{\infty} \sum_{k=0}^{n} \beta_{n,k} x_1^{(n-k)} x_2^k,\end{aligned} \tag{18}$$

where the linear stiffness coupling has been eliminated by transferring the system into the principle coordinates. The vibration-in-unison can be located by the relations between x_1 and x_2

$$x_2 = f(x_1). \tag{19}$$

Herein, if the function f is linear, the corresponding mode is called similar nonlinear normal mode; if the function f is nonlinear, the corresponding mode is called non-similar nonlinear normal mode [22]. In this study, the nonlinear terms are assumed weak, which lead to the conclusion that the linear modal motions dominate with small amendment of the orbits due to the weakly nonlinear parts.

In the next section, we study the non-gyroscopic conservative 2DOF system with repeated frequencies, which yields rich dynamic phenomena.

4. The motions of non-gyroscopic nonlinear 2DOF systems with repeated frequencies

The linear non-gyroscopic 2DOF system with repeated frequencies can be expressed as

$$\begin{aligned}\ddot{x}_1 + x_1 &= 0,\\ \ddot{x}_2 + x_2 &= 0.\end{aligned} \tag{20}$$

By choosing appropriate initial conditions, we can obtain in-unison motions and out-of-unison motions with arbitrary phase differences. For the linear case, we cannot use the term 'mode', because there are no modes since the responses of (20) are arbitrary, only depending on the initial conditions.

The nonlinear terms play key role in the modal analysis since the linear parts fail to dominate. We need a detailed analysis to demonstrate the features of the modal motions focusing on the nonlinear terms. The weakly nonlinear coupled 2DOF non-gyroscopic system with repeated frequencies, which has been studied by [23], is given by the following form

$$\begin{aligned}\ddot{x}_1 + x_1 &= -\alpha x_1^3 - \beta x_1 x_2^2,\\ \ddot{x}_2 + x_2 &= -\gamma x_2^3 - \beta x_1^2 x_2.\end{aligned} \tag{21}$$

Based on the theory of multiple scales, the fast time scale $T_0 = t$, and slow time scale $T_1 = \varepsilon t$ are introduced. The solutions to (21) are assumed as

$$\begin{aligned}x_1 &= \varepsilon x_{10} + \varepsilon^2 x_{11} + \cdots\\ x_2 &= \varepsilon x_{20} + \varepsilon^2 x_{21} + \cdots\end{aligned} \tag{22}$$

Substituting Eq. (22) and their derivatives into (21) and collecting the same orders of power in ε yield to two sets of equations

$$\varepsilon^1: \begin{cases} \dfrac{\partial^2 x_{10}}{\partial {T_0}^2} + x_{10} = 0 \\ \dfrac{\partial^2 x_{20}}{\partial {T_0}^2} + x_{20} = 0 \end{cases} \tag{23}$$

$$\varepsilon^2: \begin{cases} \dfrac{\partial^2 x_{11}}{\partial {T_0}^2} + x_{11} = -2\dfrac{\partial^2 x_{10}}{\partial T_0 \partial T_1} - \alpha x_{10}^3 - \beta x_{10} x_{20}^2 \\ \dfrac{\partial^2 x_{21}}{\partial {T_0}^2} + x_{21} = -2\dfrac{\partial^2 x_{20}}{\partial T_0 \partial T_1} - \gamma x_{20}^3 - \beta x_{10}^2 x_{20} \end{cases} \tag{24}$$

The solutions to the linear decoupled Eq. (23) can be written as

$$\begin{aligned} x_{10} &= A_1(T_1)e^{iT_0} + \overline{A}_1(T_1)e^{-iT_0} \\ x_{20} &= A_2(T_1)e^{iT_0} + \overline{A}_2(T_1)e^{-iT_0}, \end{aligned} \tag{25}$$

where

$$A_n = \frac{1}{2} a_n e^{i\theta_n}, n = 1, 2. \tag{26}$$

The real variable a_n denotes the varying amplitudes and θ_n represents the varying phase angle; both variables are functions of slow time scale T_1. Substituting Eqs. (25) and (26) into (24), and eliminating the secular terms yield

$$\begin{aligned} \frac{da_1}{dT_1} &= -\frac{\beta}{8} a_1 a_2^2 \sin 2(\theta_2 - \theta_1) \\ \frac{da_2}{dT_1} &= \frac{\beta}{8} a_1^2 a_2 \sin 2(\theta_2 - \theta_1) \\ \frac{d\theta_1}{dT_1} &= \frac{3\alpha}{8} a_1^2 + \frac{\beta}{8\omega} a_2^2 [2 + \cos 2(\theta_2 - \theta_1)] \\ \frac{d\theta_2}{dT_1} &= \frac{3\gamma}{8} a_2^2 + \frac{\beta}{8\omega} a_1^2 [2 + \cos 2(\theta_2 - \theta_1)]. \end{aligned} \tag{27}$$

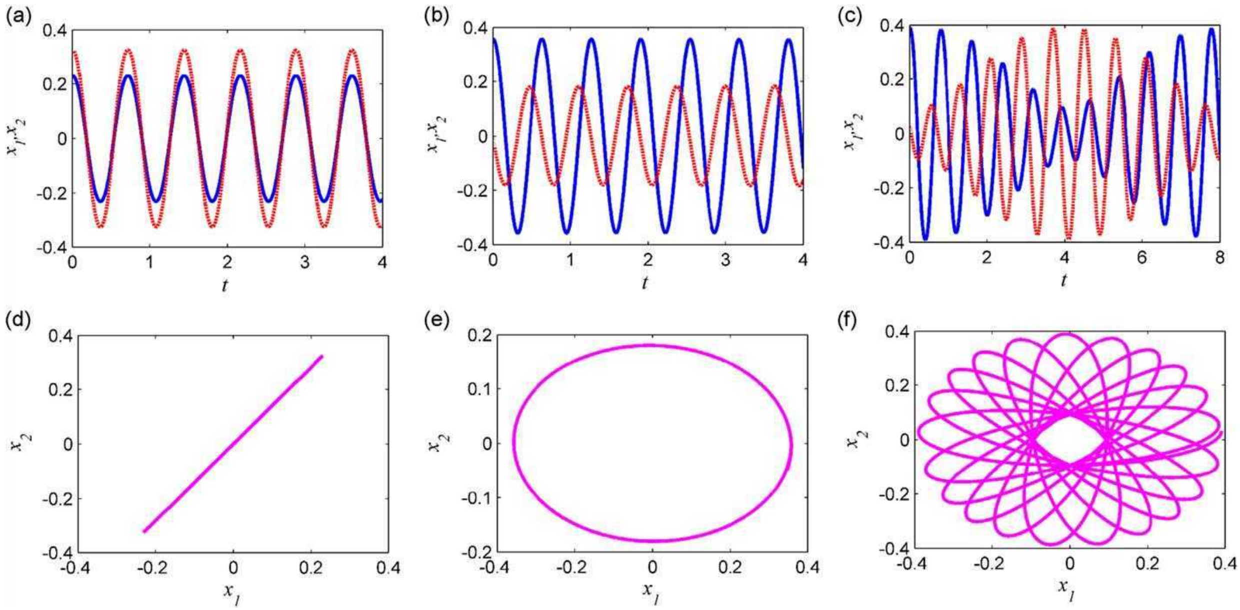

Fig. 4. The time histories and phase trajectories of nonlinear system with repeated frequencies (α= 0.14, β=0.2, γ=0.16). (a), (b), and (c) are the time histories of normal mode, elliptic mode, and transient motion, respectively; (d), (e), and (f) are the phase trajectories of normal mode, elliptic mode, and transient motion, respectively.

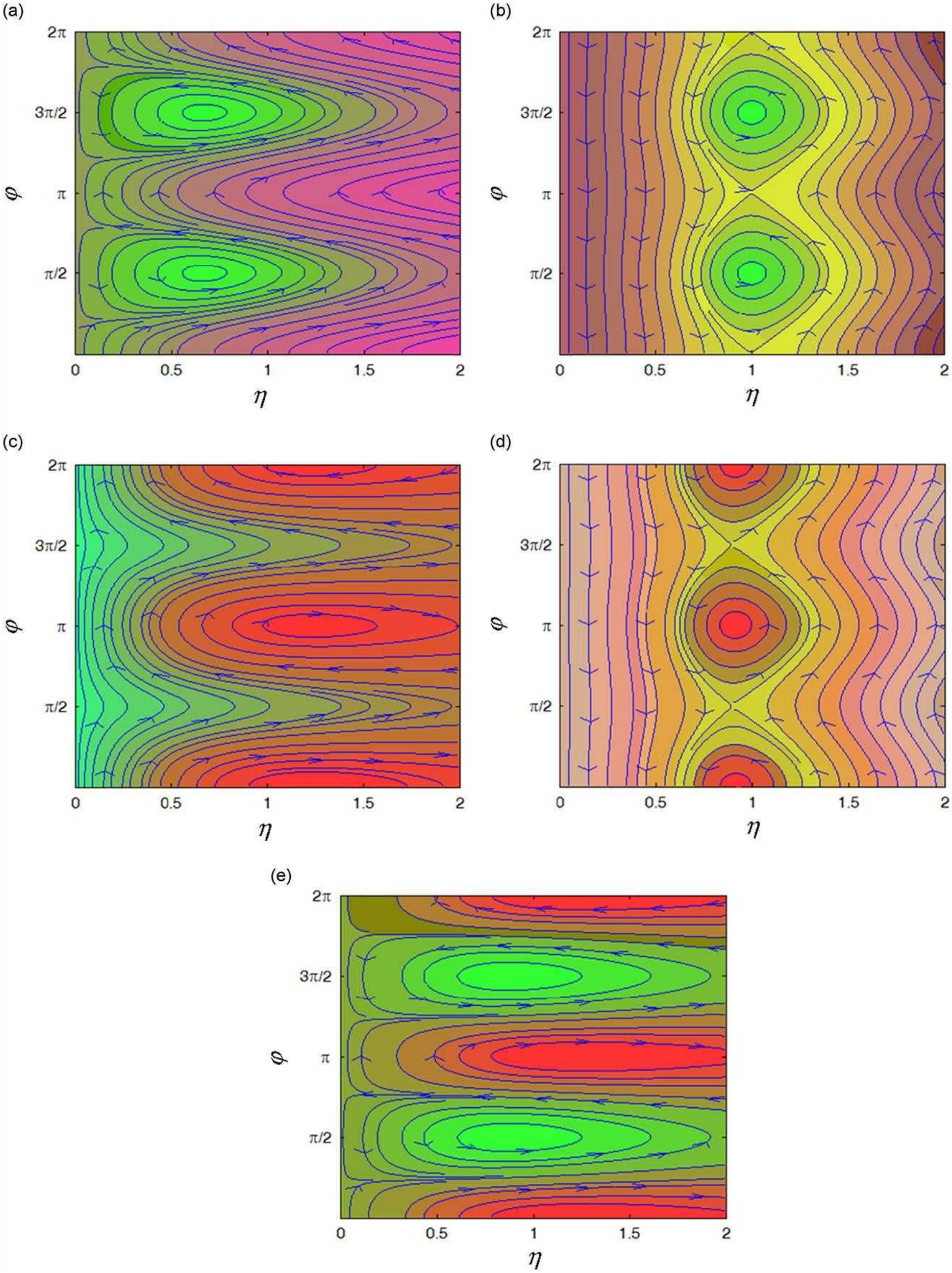

Fig. 5. Amplitude-phase portrait for all possible cases. (a) Stable EM, invalid NM ($\alpha=$ 0.15, $\beta=0.2$, $\gamma=0.25$); (b) Stable EM, unstable NM ($\alpha=0.2$, $\beta=0.03$, $\gamma=0.2$); (c) None EM, stable NM ($\alpha=$ 0.1, $\beta=0.4$, $\gamma=$ 0.2); (d) Unstable EM, stable NM ($\alpha=$ -0.4, $\beta=0.1$, $\gamma=$ -0.5); (e) Stable EM, stable NM ($\alpha=$ 0.14, $\beta=0.2$, $\gamma=0.16$).

Table 1
The comparison of different types of 2DOF nonlinear systems.

Systems	Modal motions
Gyroscopic system	2 EM
Non-gyroscopic system with distinct frequencies	2 NM
Non-gyroscopic system with repeated frequencies	2 EM, 2NM, 1EM/1NM

Since the system is conservative, we introduce the following variables as amplitude ratio, phase difference, and total energy.

$$\eta = \frac{a_2}{a_1}, \quad \varphi = \theta_2 - \theta_1, \quad E = {a_1}^2 + {a_2}^2. \tag{28}$$

By using these variables, the Eq. (27) can be transformed into

$$\begin{gathered} 8\dot{\eta} = \beta E \eta \sin 2\varphi \\ 8\dot{\varphi} = \frac{E}{1+\eta^2}\left[\left(3\gamma\eta^2 - 3\alpha\right) + \beta\left(1-\eta^2\right)\left(2 + \cos 2\varphi\right)\right], \end{gathered} \tag{29}$$

where E denotes the total energy.

The steady-state solutions of (29) can be classified into in-unison normal modes (NM) and out-of-unison elliptic modes (EM)

$$\text{NM: } \varphi = 0, \pi: \ \eta^2 = \frac{\alpha-\beta}{\gamma-\beta}; \quad \text{EM: } \varphi = \frac{1}{2}\pi, \frac{3}{2}\pi: \ \eta^2 = \frac{3\alpha-\beta}{3\gamma-\beta} \tag{30}$$

The orbits of the NM, EM, and transient motions are presented in Fig. 4. We can locate in-phase (0 phase difference) and anti-phase (π phase difference) of the normal modes, as well as the clockwise ($\pi/2$ phase difference) and anti-clockwise ($3\pi/2$ phase difference) of the elliptic mode.

The stability of the solutions of NM and EM can be determined by the eigenvalues of the linearized Jacobian of Eq. (29) and the stability conditions are

$$\text{NM: } \beta(\alpha - 2\beta + \gamma) < 0, \quad \text{EM: } \beta(3\alpha - 2\beta + 3\gamma) > 0 \tag{31}$$

Based on the existence conditions (30), and stability conditions (31), there are three cases for both the normal modes and elliptic modes, namely, nonexistent, stable, and unstable. We provide the amplitude-phase portraits for all the cases in Fig. 5 with energy $E = 40$.

Stable motions are denoted as centers, while unstable motions are presented as saddles. At phase differences 0 and π, normal modes are found, and at phase differences $\pi/2$ and $3\pi/2(-\pi/2)$, elliptic modes are detected. Since the system is conservative, solitary saddle points do not exist without centers.

5. Conclusions

In this study, three types of 2DOF systems, namely, gyroscopic system, non-gyroscopic system with distinct frequencies, and non-gyroscopic system with repeated frequencies have been discussed by the mode analysis. The gyroscopic systems only present out-of-unison modal motions (Fig. 2), the non-gyroscopic systems with distinct frequencies present in-unison modal motions (Fig. 3), while non-gyroscopic systems with repeated frequencies present both in-unison and out-of-unison modal motions (Fig. 4). The transient motions can be regarded as superposition of the modal motions for the linear case. Hence, the contour of the transient motions of gyroscopic systems are expanded by the ellipses and contour of the transient motions of non-gyroscopic systems are expanded by the linear segments.

The complex modes, based on both gyroscopic and elliptic motions, belonging to the out-of-unison motions of 2DOF oscillating systems have been analyzed, compared, and discussed. The gyroscopic motions involve two distinct frequencies with, respectively, two corresponding imaginary modes. However, the constructed weakly nonlinear 2DOF system with repeated frequencies yields both normal and elliptic modes if the existence and stability conditions are satisfied. In contrast to the two distinct frequencies of the gyroscopic motion, the elliptic motion of non-gyroscopic systems with repeated frequencies involves only single frequency with corresponding two complex modes. The features of the gyroscopic and elliptic motions are summarized in Table 1.

In this paper, we classified the planar periodic out-of-unison motions of weakly nonlinear 2DOF systems into two types, elliptic motions and gyroscopic motions, both of which are presented by elliptic orbits but with different features. The results found in this study may motivate further investigations and analysis of the complex mode of sophisticated non-normal motions existing in a wide variety of fields in science and engineering, especially in the nonlinear structures with internal resonances and in the gyrotropic motions of spin-orbital structures.

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