The Stability Condition and Energy Estimate for Nonstationary Problems of Coupled Electroelasticity

RODERICK V. NICHOLAS MELNIK
Department of Mathematics and Computing, University of Southern Queensland, QLD 4350, Australia

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Abstract: In this article, a coupled problem of dynamic electroelasticity is investigated using the variational approach and the concept of generalized solutions. The author derives a numerical procedure directly from the definition of the generalized solution of the problem and proves the convergence of the numerical scheme (with the second order in space-time) to the solution of the original problem from a class of generalized solutions. The stability condition is obtained from an energy estimate. It is shown that such a condition is the Courant-Friederichs-Lewy-type stability condition, being dependant of the velocity mixed electroelastic waves. Coupling effects are discussed with a numerical example.

1. INTRODUCTION

Modern applications of the coupled theory of dynamic electroelasticity include situations in which solutions of the underlying problems do not have to be "smooth" in a classical sense. In many practically important cases, we are dealing with steep gradients in solutions and different kinds of wave discontinuities. In such cases the original problem should be reformulated using variational principles. Such a reformulation is especially important for coupled nonstationary problems of electroelasticity. However, an approach coupling mechanical and electric fields, which is essential to obtain a plausible picture of described phenomena, involves some difficulties. These difficulties are induced primarily by the necessity to deal with the coupling phenomenon from the very beginning of the justification of the underlying mathematical model. The use of numerical procedures does not remove these difficulties. Moreover, we should be able to justify corresponding numerical procedures not only under classical smoothness assumptions[1] but also show their robustness on the classes of generalized solutions.

Studies of electromechanical interactions are important in classical areas of the mechanics of solids and in many new areas of applications in engineering and biophysics. An increasing range of applications of piezoelectrics in semiconductors and intelligent structures has stimulated a greater interest in the coupling effects between mechanical and electric fields (Carthy and Tiersten [2]; Daher [3]; Kulkarni and Hanagud [7]). For

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many of the arising problems, solutions with steep gradients or even discontinuities are typical features of underlying physical processes.

Historically, the most well-developed area of research in electroelasticity abuts to the steady-state case of harmonic oscillations. The practical importance of this case is obvious: many technical devices work in the regime of the steady-state harmonic oscillations. Nevertheless, many applications show the necessity of investigating coupled electromechanical fields that have a nonstationary rather than a steady-state character. Such problems are typical in the analysis of transient processes in various technical devices.

Certainly, in many applications, practical considerations allow us to use some simplifying assumptions in coupled electroelasticity theory. Many methods for the solution of dynamic problems in electroelasticity are based on thickness averaging[2] and the use of Kirchhoff-type hypotheses. In general, such simplifications may not be appropriate for thin structures, which are important in many applications. One of the typical examples of this type is provided by thin, hollow piezoceramic cylinders that are used as active elements in many technical devices. Furthermore, thin, hollow cylinders may provide a basis for the investigation of electromechanical processes in bones and other biological tissues.

In all such cases, as well as in many other applications, *consistent* solutions of the coupled nonstationary formulation of electroelasticity are required. Due to their generality, such applications imply that numerical methods become a natural and efficient way to find such consistent solutions. A theoretical framework for the derivation of underlying numerical procedures and their justifications is provided by the concept of generalized solutions.

This article is organized as follows:

- Section 2 concerns notation and basic preliminaries for the problem.
- In Section 3, we formulate the mathematical model and address the issues related to generalized solutions of the underlying differential problem.
- In Section 4, we derive a computational procedure using the variational approach and the definition of generalized solutions for the problem.
- The main result of Section 5 is an a priori estimate for the energy integral of the original problem.
- In Section 6, we investigate the stability of the computational model. It is shown that the stability condition can be derived from the discrete analog of the a proiri estimate obtained in Section 5.
- Section 7 provides the proof of convergence of the discrete scheme in the class of generalized solutions $W_2^4(Q_T)$ with second order with respect to space-time discretization.
- Some numerical results are presented in Section 8. Conclusions and future directions are also discussed.

2. PRELIMINARIES AND NOTATION

In the subsequent sections, we deal with a mathematical model of electroelasticity, where coupled investigation of electrical and elastic fields under nonstationary conditions is essential to obtain a plausible picture of the physical phenomena in a piezoceramic solid. We are interested in the process of coupled electroelastic nonstationary oscillations of a piezoceramic cylinder under radial preliminary polarization (Berlincourt, Curran, and Jaffe [1]; Melnik and Moskalkov [12]). Our main results concern the adequate modeling of such precesses for thin, hollow cylinders.

The following notations are used throughout the article.
Mechanical notations:

- μ denotes radial displacements of the cylinder;
- u denotes electric field potential;
- ε_r and ε_θ are components of the field of deformations;
- σ_r and σ_θ are components of the field of stresses;
- E_r denotes the radial component of electric field strength;
- D_r denotes the radial component of electric induction;
- $f_1(r,t)$ denotes a given function for the density of mass forces within the solid;
- $f_2(r,t)$ denotes a given function for electric charge density within the solid;
- e_{ij} denotes given constants of piezomoduli;
- c_{kl} denotes given constants of elastic moduli;
- ε_{11} denotes the given dielectric permittivity;
- ρ denotes the given density of the piezoceramic material;
- $V(t)$ denotes the absolute value of given potential functions on the internal and external surfaces of the cylinder[3];
- p_0 and p_1 denote given functions of stresses on the internal and external surfaces, respectively;
- $u_0(r)$ and $u_1(r)$ denote given functions of displacements and the velocity of their propagation at the initial moment of time, respectively.

Mathematical notations:

- $G = (R_0, R1)$ and $I = (0,T)$ define the range of spacial and temporal variables, respectively, where T, and R_0, R_1 are assumed to be given(R_0 and R_1 are internal and external cylinder radii); $\bar{I} = [0,T]$;

- $Q_T = I \times G$ defines the space-time region of interest; $Q_{t_1} = \{(r,t) : R_0 < r < R_1, 0 < t < t_1\}$;
- $L^2(Q_T)$ denotes the space of functions that are square integrable in $Q_T; W_2^k(Q_T)$ denotes Sobolev's spaces with an appropriate integer k ($\overset{0}{W}{}_2^k(Q_T)$ denotes the Sobolev class of functions with homogeneous boundary conditions) (Fujita and Suzuki [5]; Ladyzhenskaya [8]; Samarskii [18]; Zeidler [22]);
- M_i denotes appropriate constants in the derivation of a priori estimates (i is an integer);
- $\check{\gamma} \equiv \gamma(t-\tau)$ denotes the value of the discrete function γ on the "lower" (with respect to t) time level (τ denotes the time step of discretization);
- $\hat{\gamma} \equiv \gamma(t+\tau)$ denotes the value of the discrete function γ on the "upper" time level;
- $\gamma_{\overset{\circ}{t}} \equiv (\gamma(t+\tau) - \gamma(t-\tau))/(2\tau)$ denotes the first central difference derivative of the function γ;
- $\gamma_t \equiv (\gamma(t+\tau) - \gamma(t))/\tau$ denotes the first forward difference derivative of the function γ;
- $\gamma_{\bar{t}} \equiv (\gamma(t) - \gamma(t-\tau))/\tau$ denotes the first backward difference derivative of the function γ;
- $\gamma_{\bar{t}t} \equiv (\gamma(t+\tau) - 2\gamma(t) + \gamma(t-\tau))/\tau^2$ denotes the second central difference derivative of the function γ.[4]

Other notations are explained in the text.

3. MATHEMATICAL MODEL AND ITS GENERALIZED SOLUTION

The main results of this article are obtained for a mathematical model of dynamic electroelasticity for which the coupling effect between electrical and elastic fields is important for the adequate (both quantitative and qualitative) description of physical phenomena in a piezoceramic solid.

The process of coupled electroelastic nonstationary oscillations of a piezoceramic cylinder is modeled by a system of partial differential equations in the time-space region Q_T. The system includes the equation of motion of a continuous medium in stress and the Maxwell equation for piezoelectrics:[5]

$$(a)\ \rho \frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r) - \frac{\sigma_\theta}{r} + f_1(r,t), \quad (b)\ \frac{1}{r}\frac{\partial}{\partial r}(rD_r) = f_2(r,t). \tag{3.1}$$

The system is supplemented by initial conditions

$$u(r,0) = u_0(r), \quad \frac{\partial u(r,0)}{\partial t} = u_1(r) \tag{3.2}$$

and boundary conditions

$$\sigma_r = p_1(t), \varphi = V(t) \text{ for } r = R_0, \text{ and } \sigma_r = p_2(t), \varphi = -V(t) \text{ for } R_1. \tag{3.3}$$

For the radial preliminary polarization (strongly coupled) case, the connection between electric and elastic fields is given by the state equations

$$\begin{cases} \sigma_r = c_{11}\varepsilon_r + c_{12}\varepsilon_\theta - e_{11}E_r, \\ \sigma_\theta = c_{12}\varepsilon_r + c_{22}\varepsilon_\theta - e_{12}E_r, \\ D_r = \varepsilon_{11}E_r + e_{12}\varepsilon_\theta + e_{11}\varepsilon_r, \end{cases} \tag{3.4}$$

where the relationship between deformations and displacements are given in the Cauchy form, and potential is introduced in its electrostatic form as

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r}, \quad E_r = -\frac{\partial \varphi}{\partial r}. \tag{3.5}$$

We assume nonnegativeness for the potential energy of deformation—that is, $\exists\, \delta > 0$ such that $\forall\, \xi_1, \xi_2$,

$$\delta\left(\xi_1^2 + \xi_2^2\right) \le c_{11}\xi_1^2 + 2c_{12}\xi_1\xi_2 + c_{22}\xi_2^2. \tag{3.6}$$

Solutions of such nonstationary problems of coupled electroelasticity are of great importance for the reliable modeling of many technical devices, such as piezovibrators, several different types of transmitters, generators, and so on ([1]; Melnik and Moskalkov [13]; Nowacki [17]). Other applications arise in different areas of engineering, such as hydrodynamics, and in biophysics.

Rigorous mathematical investigation of the model involves difficulties caused by the coupling effect. When we rewrite (3.1) using (3.4) and (3.5), we have not only a strongly coupled system of partial differential equations[6] that consists of two types of operators (hyperbolic and elliptic),

$$\begin{cases} \rho\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}c_{11}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - c_{22}\frac{u}{r^2} + \frac{1}{r}e_{11}\frac{\partial^2 \varphi}{\partial r^2} - \frac{1}{r}e_{12}\frac{\partial \varphi}{\partial r} + f_1(r,t), \\ -\varepsilon_{11}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + e_{11}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + e_{12}\frac{1}{r}\frac{\partial u}{\partial r} = f_2(r,t), \end{cases}$$

but also a strong coupling effect on the boundary through the boundary conditions for stresses in (3.3). Under simultaneous solution of the electroelasticity system, both elastic and electroelastic nonlinearities in piezoelectrics are small[7]; thus, the model (3.1) through (3.5) provides an adequate description of the underlying physical processes. The correctness of this model was investigated in Melnik [11] and [13].

In what follows, we develop a technique that allows us to derive efficient computational procedures for the investigation of coupling effects in dynamic electroelasticity problems. We are especially interested in cases where coupling effects manifest themselves significantly, having a decisive influence on the output characteristics of the designed devices and feedback mechanisms. We shall give a rigorous basis for the derived computational procedure, which can give practically acceptable results even if the solution is not required to be smooth. A general framework for our approach gives the concept of generalized solutions.

According to the general approach to the problems of mathematical physics ([5]; [8]; Samarskii, Lazarov, and Makarov [19]; [22]), we introduce the following definition.

Definition 3.1. *We call a pair of functions*

$$(u(r,t),\phi(r,t)) \in W_2^1(Q_T) \times L^2\left(I, \overset{0}{W}{}_2^1(G)\right)$$

($u(r,t)$ equals $u_0(r)$ for $t=0$) a generalized solution of the coupled problem of dynamic electroelasticity (3.1) through (3.5) if it satisfies the following integral identities:

$$\int_{Q_T} r\left(-\rho\frac{\partial u}{\partial t}\frac{\partial \eta}{\partial t} + \sigma_r\frac{\partial \eta}{\partial r} + \frac{\sigma_\theta}{r}\eta\right)drdt - \int_{R_0}^{R_1} r\rho u_1(r)\eta(r,0)dr$$
$$= \int_{Q_T} rf_1\eta drdt \quad \forall \eta \in \hat{W}_2^1(Q_T), \qquad (3.7)$$

$$\int_{R_0}^{R_1}\left(\varepsilon_{11}r\frac{\partial\phi}{\partial r}\frac{\partial\zeta}{\partial r} + e_{11}r\varepsilon_r\frac{\partial\zeta}{\partial r} + e_{12}r\varepsilon_\theta\frac{\partial\zeta}{\partial r}\right)dr = \int_{R_0}^{R_1} rf_2\zeta dr \quad \forall\zeta \in W_2^1(G), \qquad (3.8)$$

almost everywhere in I. Here, $\hat{W}_2^1(Q_T)$ stands for the subspace of $W_2^1(Q_T)$ that consists of all elements of $W_2^1(Q^T)$ that equal zero when $t=T$ (for simplicity, we set $p_i(t) = V(t) = 0$, $i = 1,2$ in the boundary conditions(3.3)).

The differential equations of electroelasticity are a partial case of a more general variational formulation (Mindlin [14]; Yang and Batra [20]) induced by the concept of generalized solutions and physical principles. Moreover, the set of differential equations (3.1) through (3.5) may be obtained from such formulations only under certain (usually excessive) smoothness assumptions. One way to do this is to equate the first variation of the Lagrangian of the electromechanical system to zero or, alternatively, to use different forms of conservation laws to obtain differential forms of the equations. The a priori smoothness for the solution assumed with this reasoning is often questionable in practical applications [8, 19, 22].

Therefore, to have robust numerical procedures, it is reasonable to derive computational models directly from Definition 3.1 with generalized solutions. If a discrete model is obtained by an appropriate approximation of the variational functional that involves the energy of the system, the corresponding discrete model will ensure robustness for nonsmooth solutions of the problem.[8]

4. NUMERICAL PROCEDURE FOR FINDING GENERALIZED SOLUTIONS

Let us assume that generalized second derivatives of the solution are square integrable functions from L^2 (existence and uniqueness of such generalized solutions was proved in [11]). Then, the solution $(u(r,t), \phi(r,t))$ satisfies the initial system (3.1) through (3.5) in the sense of the integral identities (3.7), (3.8) and the integral identity

$$\int_{Q_T} r\left(\rho\frac{\partial^2 u}{\partial t^2}\eta + \sigma_r\frac{\partial \eta}{\partial r} + \frac{\sigma_\theta}{r}\eta\right) drdt = \int_{Q_T} r f_1 \eta drdt, \tag{4.1}$$

where η is an arbitrary element from $W_2^{1,0}(Q_T0)$.[9] Choosing in (4.1) the function $\eta(r,t)$ of the form

$$\eta(r,t) \equiv \begin{cases} 0 & \text{for } t \in [t_1, T], \\ \frac{\partial u}{\partial t} & \text{for } t \in [0, t_1), \end{cases}$$

and taking into consideration that[10]

$$\int_{Q_T} \left(\sigma_r\frac{\partial_{\varepsilon_r}}{\partial t} + \sigma_\theta\frac{\partial_{\varepsilon_\theta}}{\partial t}\right) drdt = \int_{Q_T} \left[c_{11}\varepsilon_r\frac{\partial_{\varepsilon_r}}{\partial t} + c_{12}\left(\frac{\partial_{\varepsilon_r}}{\partial t}\varepsilon_\theta + \frac{\partial_{\varepsilon_\theta}}{\partial t}\varepsilon_r\right)\right.$$
$$\left. + c_{22}\varepsilon_\theta\frac{\partial_{\varepsilon_\theta}}{\partial t} + \varepsilon_{11}\frac{\partial E_r}{\partial t}E_r - \frac{\partial D_r}{\partial t}E_r\right] drdt.$$

we obtain the following integral equality to characterize energy ($\mathcal{E}$) change in the electromechanical system:

$$\int_0^{t_1} \frac{d\mathcal{E}}{dt}dt = \int_{Q_{t_1}} r f_1 \frac{\partial u}{\partial t} drdt + \int_{Q_{t_1}} r\frac{\partial D_r}{\partial t}E_r drdt + \int_0^{t_1}\left[R_1 p_1\frac{\partial u(R_1,t)}{\partial t} - R_0 p_0\frac{\partial u(R_0,t)}{\partial t}\right] dt. \tag{4.2}$$

The total energy of the electromechanical system $\mathcal{E}$ can be written as a sum $\mathcal{E} = K + W + P$, where

$$K = \frac{\rho}{2}\int_{R_0}^{R_1} r\left(\frac{\partial u}{\partial t}\right)^2 dr$$

is the kinetic energy,

$$W = \frac{1}{2}\int_{R_0}^{R_1} r\left[c_{11}\varepsilon_r^2 + 2c_{12}\varepsilon_r\varepsilon_\theta + c_{22}\varepsilon_\theta^2\right] dr$$

is the energy of elastic deformation, and

$$P = \frac{\varepsilon_{11}}{2}\int_{R_0}^{R_1} rE_r^2$$

is the energy of electric field of the system.

To find the integral $\int_{Q_{t_1}} r(\partial D_r/\partial t)E_r drdt$, we use identity (3.8) integrated in t from 0 to t_1, setting in it

$$\zeta(r,t) = \begin{cases} 0 & \text{for } t \in [t_1, T], \\ \frac{\partial\phi}{\partial t} & \text{for } t \in [0, t_1). \end{cases}$$

After a simple transformation, we have[11]

$$-\int_{Q_{t_1}} r\frac{\partial D_r}{\partial t}E_r drdt = -\int_{Q_{t_1}} r\frac{\partial f_2}{\partial t}\phi drdt + \int_0^{t_1} r\,\frac{\partial D_r}{\partial t}\phi\Big|_{R_0}^{R_1} dt. \tag{4.3}$$

Taking into consideration the fact that identities (4.2) and (4.3) are satisfied for any $t_1 \in \bar{I}$, we obtain the energy balance identity for a peizoelectric solid:

$$\frac{d\mathcal{E}}{dt} = \left[R_1 p_1 \frac{\partial u(R_1,t)}{\partial t} - R_0 p_0 \frac{\partial u(R_0,t)}{\partial t}\right] + \int_{R_0}^{R_1} rf_1\frac{\partial u}{\partial t}dr$$
$$+ \int_{R_0}^{R_1} r\phi\frac{\partial f_2}{\partial t}dr + V(t)\left[\frac{\partial D_r(R_1,t)}{\partial t}R_1 + \frac{\partial D_r(R_0,t)}{\partial t}R_0\right]. \tag{4.4}$$

The right-hand side of (4.4) contains those sources that cause dynamic behavior (i.e., loads on the surface of the body), mass forces, and surface charges. It is easy to see that from the relationship (4.4), we can obtain the equation of motion (3.1a), the Maxwell equation (3.1b), and the *natural* boundary conditions of the problem (3.1) through (3.5).

Let us now derive a discrete version of (4.4). We introduce a difference grid covering the region Q_T,

$$\overline{\omega}_{h_\tau} = \overline{\omega}_h \times \overline{\omega}_\tau,$$

where

$$\overline{\omega}_h = \left\{ r_i = R_0 + ih,\ h = \frac{R_1 - R_0}{N},\ i = 0,1,\ldots,N \right\}, \text{ and}$$

$$\overline{\omega}_\tau = \{ t_j = j\tau,\ \tau = T/L,\ j = 0,1,\ldots,L \}.$$

Let y and μ be the functions of two discrete variables defined on this grid that approximate displacement $u(r,t)$ and electrostatic potential $\varphi(r,t)$, respectively. For each $t \in \overline{\omega}_\tau$, these functions are elements of the Hilbert spaces

$$H_1 = \{ y(r) : r \in \overline{\omega}_h \}, \quad H_2^0 = \{ \mu(r) : r \in \omega_h; \mu = 0, r = R_0, R_1 \},$$

with the scalar product $(y,v) = \sum_{\overline{\omega}_h} \hbar yv$, where $\hbar = h/2$ for $i = 0,N$, and $\hbar = h$ for $i = 1,\ldots,N-1$. Also, let

$$\omega_h^+ = \left\{ r_i = R_0 + ih,\ i = \overline{1,N} \right\},\ \omega_h^- = \{ r_i = R_0 + ih,\ i = 0,1,\ldots,N-1 \}.$$

We shall derive our computational model in two stages. First, we approximate the integral of kinetic energy by the composite trapezoidal rule in the space variable r, that is,[12]

$$K^h = \frac{\rho}{2} \sum_{\overline{\omega}_h} \hbar r \left(\frac{d\widetilde{u}}{dt} \right)^2, \text{ where } K = K^h + O(h^2),$$

whereas the integrals of elastic deformation and electric field are approximated by the composite rectangular rule

$$W^h + P^h = \frac{1}{2} \sum_{\omega_h^+} h\bar{r} \left[c_{11}\widetilde{\varepsilon}_r^2 + 2c_{12}\widetilde{\varepsilon}_r\widetilde{\varepsilon}_\theta + c_{22}\widetilde{\varepsilon}_\theta^2 + \varepsilon_{11}\widetilde{E}_r^2 \right],$$

where

$$W + P = W^h + P^h + O(h^2),$$

and

$$\widetilde{\varepsilon}_r = \widetilde{u}_{\bar{r}},\ \widetilde{\varepsilon}_\theta = \left(\widetilde{u} + \widetilde{u}^{(-1)} \right) / (2\bar{r}),\ \widetilde{E}_r = -\widetilde{\phi}_{\bar{r}},\ \widetilde{u}^{(\pm 1)} = \widetilde{u}(r \pm h, t),\ \bar{r} = r - h/2r, \in \omega_h.$$

Then, we approximate the left-hand side of (4.4) as

$$\frac{d\widetilde{\mathcal{E}}}{dt} = \rho \sum_{\overline{\omega}_h} \hbar r \widetilde{v} \frac{d\widetilde{v}}{dt} + \sum_{\omega_h^+} h\bar{r} \left[\frac{\partial \widetilde{\varepsilon}_r}{\partial t} \widetilde{\sigma}_r + \frac{\partial \widetilde{\varepsilon}_\theta}{\partial t} \widetilde{\sigma}_\theta \right] + \sum_{\omega_h^+} h\bar{r} \widetilde{E}_r \frac{\partial \widetilde{D}_r}{\partial t},$$

where $\widetilde{\mathcal{E}}$ is a differential-difference analog of the total energy of the electromechanical system, $\widetilde{v} = d\widetilde{u}/dt$, $\widetilde{\sigma}_r = c_{11}\widetilde{\varepsilon}_r + c_{12}\widetilde{\varepsilon}_\theta - e_{11}\widetilde{E}_r$, and $\widetilde{\sigma}_\theta = c_{12}\widetilde{\varepsilon}_r + c_{22}\widetilde{\varepsilon}_\theta - e_{12}\widetilde{E}_r$. Now, after simple transformations,[13] we obtain a differential-difference analog of the energy identity (4.4) as

$$\rho\sum_{\overline{\omega}_h}\hbar r\widetilde{v}\frac{d\widetilde{v}}{dt}-\sum_{\omega_h}r\widetilde{v}h\frac{1}{r}(\overline{r}\widetilde{\sigma}_r)_r+\sum_{\omega_h^+}rh\widetilde{v}\frac{\widetilde{\sigma}_\theta}{2r}+\sum_{\omega_h^-}rh\widetilde{v}\frac{\widetilde{\sigma}_\theta^{(+1)}}{2r}+\sum_{\omega_h^+}\overline{r}h\widetilde{E}_r\frac{\partial\widetilde{D}_r}{\partial t}$$

$$=[R_1p_1\widetilde{v}(R_1,t)-R_0p_0\widetilde{v}(R_0,t)]-\widetilde{v}_N\overline{r}_N(\widetilde{\sigma}_r)_N+\widetilde{v}_0\overline{r}_1(\widetilde{\sigma}_r)_1+\sum_{\overline{\omega}_h}\hbar r\widetilde{v}f_1+\sum_{\omega_h}rh\widetilde{\phi}\frac{\partial f_2}{\partial t}$$

$$+V(t)\left[\overline{R}_1\frac{\partial\widetilde{D}_r(\overline{R}_1,t)}{\partial t}+\overline{R}_0^{(+1)}\frac{\partial\widetilde{D}_r(\overline{R}_0^{(+1)},t)}{\partial t}\right]. \tag{4.5}$$

- Assuming $\widetilde{v}$ is not identically zero when $\partial\widetilde{D}_r/\partial t=\partial f_2/\partial t=0$ from (4.5), we can derive a differential-difference analog of the equation for continuous medium motion and boundary conditions for stresses in (3.3) (since the latter are natural boundary conditions).
- On the other hand, assuming that $\partial\widetilde{D}_r/\partial t$ is not identically zero when $\widetilde{v}=f_1=0$, we can obtain a differential-difference analog of Maxwell equation for piezoelectrics as

  $$\sum_{\omega_h^+}\overline{r}h\widetilde{E}_r\frac{\partial\widetilde{D}_r}{\partial t}=\sum_{\omega_h}rh\widetilde{\phi}\frac{\partial f_2}{\partial t}+V(t)\left[\overline{R}_1\frac{\partial\widetilde{D}_r\left(\overline{R}_1,t\right)}{\partial t}+\overline{R}_0^{(+1)}\frac{\partial\widetilde{D}_r\left(\overline{R}_0^{(+1)},t\right)}{\partial t}\right].$$

  If we take into consideration that $\widetilde{E}_r=-\widetilde{\phi}_{\overline{r}}$ and set[14]

  $$\widetilde{\phi}=V(t)\text{ when }r=R_0,\text{ and }\widetilde{\phi}=-V(t)\text{ when }r=R_1,$$

  then the differential-difference analog of the Maxwell equation can be rewritten as

  $$\left(\widetilde{\phi},\left(\overline{r}\frac{\partial\widetilde{D}_r}{\partial t}\right)_r\right)=\left(\widetilde{\phi},r\frac{\partial f_2}{\partial t}\right).$$

The second stage of our derivation consists of time discretization of the differential-difference scheme obtained with the first stage. Finally, we obtain the discrete space-time scheme for the solution of the problem (3.1) through (3.5), which consists of

- the approximation of the equation of motion and boundary conditions for stresses for $t\in\overline{\omega}_\tau$:

$$\rho y_{\bar{t}t}=\begin{cases}\frac{1}{r}\left(\overline{r}\overline{\sigma}_r\right)_r-\frac{\overline{\sigma}_\theta^{(+1)}+\overline{\sigma}_\theta}{2r}+f_1 & \text{for } r\in\omega_h,\\ \frac{2}{h}\frac{1}{r}\overline{r}^{(+1)}\overline{\sigma}_r^{(+1)}-\frac{\overline{\sigma}_\theta^{(+1)}}{r}+f_1-\frac{2}{h}p_0 & \text{for } r=R_0,\\ -\frac{2}{h}\frac{1}{r}\overline{r}\overline{\sigma}_r-\frac{\overline{\sigma}_\theta}{r}+f_1+\frac{2}{h}p_1 & \text{for } r=R_1;\end{cases} \tag{4.6}$$

- the approximation of the Maxwell equation for piezoelectrics and the relationship for electric potential:

$$\frac{1}{r}(\bar{r}\overline{D}_r)_r = f_2, \quad \overline{E}_r = -\mu_{\bar{r}}; \tag{4.7}$$

- the approximation of the state equations:

$$\begin{cases} \bar{\sigma}_r = c_{11}\bar{\varepsilon}_r + c_{12}\bar{\varepsilon}_\theta - e_{11}\overline{E}_r, \\ \bar{\sigma}_\theta = c_{12}\bar{\varepsilon}_r + c_{22}\bar{\varepsilon}_\theta - e_{12}\overline{E}_r, \\ \overline{D}_r = \varepsilon_{11}\overline{E}_r + e_{12}\bar{\varepsilon}_\theta + e_{11}\bar{\varepsilon}_r, \end{cases} \tag{4.8}$$

where we approximate the Cauchy relations as

$$\bar{\varepsilon}_r = (y - y^{(-1)})/h, \quad \bar{\varepsilon}_\theta = (y + y^{(-1)})/(2\bar{r}); \tag{4.9}$$

- the exact boundary conditions for the potential function:

$$\mu = V(t) \text{ for } r = R_0, \text{ and } \mu = -V(t) \text{ for } r = R_1; \tag{4.10}$$

- the first initial condition:

$$y(r,0) = u_0(r); \tag{4.11}$$

- and the second initial condition approximated by the central difference derivative with subsequent elimination of the fictitious time layer for $t = 0$, that is,

$$\rho y_t = \rho u_1(r) + \frac{\tau}{2}\begin{cases} \frac{1}{r}(\bar{r}\bar{\sigma}_r)_r - \frac{\bar{\sigma}_\theta^{(+1)} + \bar{\sigma}_\theta}{2r} + f_1 & \text{for } r \in \omega_h, \\ \frac{2}{h}\frac{1}{r}\bar{r}^{(+1)}\bar{\sigma}_r^{(+1)} - \frac{\bar{\sigma}_\theta^{(+1)}}{r} + f_1 - \frac{2}{h}p_0 & \text{for } r = R_0, \\ -\frac{2}{h}\frac{1}{r}\bar{r}\bar{\sigma}_r - \frac{\bar{\sigma}_\theta}{r} + f_1 + \frac{2}{h}p_1 & \text{for } r = R_1. \end{cases} \tag{4.12}$$

Investigation of the stability of this discretized scheme can be performed using a difference analog of the energy identity (4.4). The stability condition for the scheme is provided by the requirement of nonnegativeness for the difference analog of the energy integral. Of course, the rate of convergence of such a scheme is virtually defined by the order of approximation, which depends on a priori assumption for the solution smoothness. Since in many practical applications the solution does not possess continuous derivatives, to justify the computational model, we should investigate its convergence in classes of generalized solutions.

5. AN A PROIRI ESTIMATE FOR THE ENERGY INTEGRAL

Mathematically speaking, with the condition (3.6), we can obtain an a priori estimate for the solution of the problem (3.1) through (3.5) from the energy balance identity (4.4).

It should be noted, however, that the presence of time derivatives of electric induction in (4.4) indicates that, in general, the equation of forced electrostatics (3.1b) should be supplemented by the equation

$$\nabla \times H = \frac{1}{c_l}\frac{\partial D_r}{\partial t},$$

where H is the strength of the magnetic field, and c_l is the velocity of light [9]. We assume, however, that[15]

$$\frac{\partial D_r}{\partial t} = 0. \tag{5.1}$$

Then the energy balance equation (4.4) can be rewritten in the form

$$\frac{d\mathcal{E}}{dt} = \left[R_1 p_1 \frac{\partial u(R_1,t)}{\partial t} - R_0 p_0 \frac{\partial u(R_0,t)}{\partial t}\right] + \int_{R_0}^{R_1} r f_1 \frac{\partial u}{\partial t} dr. \tag{5.2}$$

Integrating (5.2) in t from zero to a certain $t_1 (0 \leq t_1 \leq T)$, we obtain

$$\mathcal{E}(t_1) = \mathcal{E}(0) + \int_0^{t_1} [R_1 p_1 v_N - R_0 p_0 v_0]\, dt + \int_0^{t_1}\int_{R_0}^{R_1} r f_1 \frac{\partial u}{\partial t} drdt. \tag{5.3}$$

Performing estimates of terms on the right-hand side of (5.3), using the Cauchy-Schwarz inequality and embedding theorems [5, 8, 19, 22][16] after simple but cumbersome calculations, we obtain

$$\mathcal{E}(t_1) \leq M_1\mathcal{E}(0) + M_2 \sum_{i=0}^{1} \left[|p_i(t_1)|^2 + |p_i(0)|^2\right]$$
$$+ M_3 \int_0^{t_1} \left[\left(\frac{\partial p_0}{\partial t}\right)^2 + \left(\frac{\partial p_1}{\partial t}\right)^2\right] dt + M_4 \int_0^{t_1} \mathcal{E}(t)dt + M_5 \int_0^{t_1}\int_{R_0}^{R_1} r f_1^2 drdt.$$

Using the lemma on integral inequality [8], we get from the last expression

$$\mathcal{E}(t_1) \leq M_6\mathcal{E}(0) + M_2 \sum_{i=0}^{1} \left[|p_i(t_1)|^2 + |p_i(0)|^2\right]$$
$$+ M_3 \int_0^{t_1} \left[\left(\frac{\partial p_0}{\partial t}\right)^2 + \left(\frac{\partial p_1}{\partial t}\right)^2\right] dt + M_5 \int_0^{t_1}\int_{R_0}^{R_1} r f_1^2 drdt. \tag{5.4}$$

Estimation of the total energy of the electromechanical system at the initial moment of time $\mathcal{E}(0)$ is made by taking into consideration the initial conditions of the problem. The main difficulty is estimating the functional $\int_{R_0}^{R_1} r E_r^2|_{t=0} dr$. This functional can be estimated using the Maxwell equation written formally for $t = 0$, multiplied by $r\phi$ and integrated in r for R_0 to R_1. As a result, taking into consideration (3.6) and (5.4), we have the a priori estimate

$$\varepsilon(t_1) \leq M \left\{ \rho \int_{R_0}^{R_1} r u_1^2 dr + \int_{R_0}^{R_1} r \left(c_{11}\varepsilon_r^2 + 2c_{12}\varepsilon_r\varepsilon_\theta + c_{22}\varepsilon_\theta^2 \right)\big|_{t=0} dr \right.$$
$$+ |V(0)|^2 + \sum_{i=0}^{1} \left[|p_i(t_1)|^2 + |p_i(0)|^2 \right] + \int_0^{t_1} \left[\left(\frac{\partial p_0}{\partial t} \right)^2 + \left(\frac{\partial p_1}{\partial t} \right)^2 \right] dt$$
$$\left. + \int_{R_0}^{R_0} r\lambda^2|_{t=0} dr + \int_0^{t_1}\int_{R_0}^{R_1} r f_1^2 dr dt \right\}, \tag{5.5}$$

where λ is defined from $rf_2 = \partial(r\lambda)/\partial r$, $\lambda|_{r=R_0} = 0$. We formulate the final result in the following lemma.

Lemma 5.1. *If the condition (3.6) holds, then the solution of the coupled problem of dynamic electroelasticity (3.1) through (3.5) satisfies the energy inequality (5.5) for all $t_1 \in (0, T]$ and a certain constant $M > 0$. The energy integral (5.5) is defined as*

$$\varepsilon(t) = \frac{\rho}{2} \int_{R_0}^{R_1} r \left(\frac{\partial u}{\partial t} \right)^2 dr + \frac{1}{2} \int_{R_0}^{R_1} r \left[c_{11}\varepsilon_r^2 + 2c_{12}\varepsilon_r\varepsilon_\theta + c_{22}\varepsilon_\theta^2 \right] dr + \frac{\varepsilon_{11}}{2} \int_{R_0}^{R_1} r E_r^2.$$

Lemma 5.1 is a key factor in establishing a stability condition for the discretized problem (4.6) through (4.12).

6. ANALYSIS OF THE STABILITY OF THE DISCRETIZED SCHEME

The main result to be established in this section is a discrete analog of (5.5). To obtain this result, we should first derive an analog of the energy balance equation (5.2) for the discrete problem (4.6) through (4.12).

Let us take a scalar product between the equation (4.6) ($r \in \omega_h$) and the discrete function $2\tau r y_{\mathring{t}}$. We sum the result ($\forall r \in (R_0, R_1)$) in i form 1 to $N-1$ and use (4.6) written for $r = R_0, R_1$, which give approximations to boundary conditions for stresses

$$\frac{h}{2}(r2\tau v)\rho y_{\bar{t}t} = (2\tau r v)\frac{1}{r}(\overline{r\sigma}_r)^{(+1)} - \frac{h}{2}(2\tau r v)\frac{\overline{\sigma}_\theta^{(+1)}}{r} + \frac{h}{2}(2\tau r v) f_1 - (2\tau r v) p_0, \quad r = R_0$$
$$\frac{h}{2}(r2\tau v)\rho y_{\bar{t}t} = (2\tau r v)\frac{1}{r}(\overline{r\sigma}_r) - \frac{h}{2}(2\tau r v)\frac{\overline{\sigma}_\theta}{r} + \frac{h}{2}(2\tau r v) f_1 - (2\tau r v) p_1, \quad r = R_1$$

where for simplicity we denote $v = y_{\mathring{t}}$.

Then, taking into consideration the easily verified identities

$$2\tau\sum_{\omega_h}\rho y_{\bar{t}t}vrh+2\tau\left(\rho y_{\bar{t}t}vr\frac{h}{2}\right)\Big|_{R_0}+2\tau\left(\rho y_{\bar{t}t}vr\frac{h}{2}\right)\Big|_{R_1}=\sum_{\overline{\omega}_h}\hbar r\rho(y_t-y_{\bar{t}})(y_t+y_{\bar{t}}),$$

$$2\tau\left[-\sum_{\omega_h}rhv\frac{1}{r}(\bar{r}\overline{\sigma}_r)_r+v_N\bar{r}_N(\overline{\sigma}_r)_N-v_0\bar{r}_1(\overline{\sigma}_r)_1\right]=2\tau\sum_{\omega_h^+}\bar{r}h\overline{\sigma}_r(\overline{\varepsilon}_r)_{\check{t}},$$

$$2\tau\left[\sum_{\omega_h}rhv\frac{\overline{\sigma}_\theta^{(+1)}+\overline{\sigma}_\theta^{(+1)}}{2r}+\frac{h}{2}\left(rv\frac{\overline{\sigma}_\theta^{(+1)}}{r}\right)\Big|_{R_0}+\frac{h}{2}\left(rv\frac{\overline{\sigma}_\theta}{r}\right)\Big|_{R_1}\right]$$

$$=2\tau\left[\sum_{\omega_h^+}\bar{r}h\frac{\overline{\sigma}_\theta}{2\bar{r}}v+\sum_{\omega_h^-}\bar{r}^{(+1)}h\frac{\overline{\sigma}_\theta^{(+1)}}{2\bar{r}^{(+1)}}v\right]=2\tau\sum_{\omega_h^+}\bar{r}h\overline{\sigma}_\theta\left(\overline{\varepsilon}_\theta\right)_{\check{t}},$$

we get

$$\sum_{\overline{\omega}_h}\hbar r\rho(y_t-y_{\bar{t}})(y_t+y_{\bar{t}})+2\tau\sum_{\omega_h^+}h\bar{r}\left[\overline{\sigma}_r(\overline{\varepsilon}_r)_{\check{t}}+\overline{\sigma}_\theta(\overline{\varepsilon}_\theta)_{\check{t}}\right]$$

$$=2\tau\left[\sum_{\overline{\omega}_h}\hbar r y_{\check{t}}f_1+\left(R_1p_1y_{\check{t}}|_{R_1}-R_0p_0y_{\check{t}}|_{R_0}\right)\right]. \tag{6.1}$$

According to the approximation of state equations (4.8), we have

$$\overline{\sigma}_r\left(\overline{\varepsilon}_r\right)_{\check{t}}+\overline{\sigma}_\theta\left(\overline{\varepsilon}_\theta\right)_{\check{t}}=c_{11}\overline{\varepsilon}_r\left(\overline{\varepsilon}_r\right)_{\check{t}}+c_{12}\left[\overline{\varepsilon}_\theta\left(\overline{\varepsilon}_r\right)_{\check{t}}+\left(\overline{\varepsilon}_\theta\right)_{\check{t}}\overline{\varepsilon}_r\right]$$
$$+c_{22}\overline{\varepsilon}_\theta\left(\overline{\varepsilon}_\theta\right)_{\check{t}}+\varepsilon_{11}\overline{E}_r\left(\overline{E}_r\right)_{\check{t}}-\left(\overline{D}_r\right)_{\check{t}}\overline{E}_r.$$

Also, we can easily verify that

$$2\tau\sum_{\omega_h^+}h\bar{r}\overline{\varepsilon}_r\left(\overline{\varepsilon}_r\right)_{\check{t}}=\mathcal{A}-\frac{\tau^2}{4}\quad\mathcal{B},\text{ and }2\tau\sum_{\omega_h^+}h\bar{r}\left[\overline{\varepsilon}_\theta\left(\overline{\varepsilon}_r\right)_{\check{t}}+\left(\overline{\varepsilon}_\theta\right)_{\check{t}}\overline{\varepsilon}_r\right]=I\left(t+\tau\right)-I(t),$$

where

$$\mathcal{A}=\frac{1}{4}\sum_{\omega_h^+}h\bar{r}(\hat{\overline{\varepsilon}}_r+\overline{\varepsilon}_r)^2-\frac{1}{4}\sum_{\omega_h^+}h\bar{r}(\overline{\varepsilon}_r+\check{\overline{\varepsilon}}_r)^2,\quad\mathcal{B}=\sum_{\omega_h^+}h\bar{r}((\overline{\varepsilon}_r)_t)^2-\sum_{\omega_h^+}h\bar{r}((\overline{\varepsilon}_r)_{\bar{t}})^2,$$

$$I(t)=\sum_{\omega_h^+}h\bar{r}\left[\overline{\varepsilon}_r\overline{\varepsilon}_\theta+\check{\overline{\varepsilon}}_r\check{\overline{\varepsilon}}_\theta-\tau^2\left(\overline{\varepsilon}_r\right)_{\bar{t}}\left(\overline{\varepsilon}_\theta\right)_{\bar{t}}\right].$$

Then, setting[17] $\left(\overline{D}_r\right)_{\bar{t}}=0$ and introducing the discrete analog of the total energy of the electromechanical system as

$$\overline{\varepsilon}(t) = \rho \sum_{\overline{\omega}_h} \hbar r y_{\bar{t}}^2 + \sum_{\omega_h^+} h\bar{r} \left\{ c_{11} \left[\frac{1}{4} \left(\bar{\varepsilon}_r + \check{\bar{\varepsilon}}_r \right)^2 - \frac{\tau^2}{4} ((\bar{\varepsilon}_r)_{\bar{t}})^2 \right] \right.$$
$$+ c_{12} \left[\bar{\varepsilon}_r \bar{\varepsilon}_\theta + \check{\bar{\varepsilon}}_r \check{\bar{\varepsilon}}_\theta - \tau^2 (\bar{\varepsilon}_r)_{\bar{t}} (\bar{\varepsilon}_\theta)_{\bar{t}} \right] + c_{22} \left. \left[\frac{1}{4} \left(\bar{\varepsilon}_\theta + \check{\bar{\varepsilon}}_\theta \right)^2 - \frac{\tau^2}{4} ((\bar{\varepsilon}_\theta)_{\bar{t}})^2 \right] \right\}$$
$$+ \sum_{\omega_h^+} h\bar{r}\varepsilon_{11} \left[\frac{1}{4} \left(\overline{E}_r + \check{\overline{E}}_r \right)^2 - \frac{\tau^2}{4} ((\overline{E}_r)_{\bar{t}})^2 \right], \tag{6.2}$$

we get the discrete analog of the energy identity (5.2)

$$\overline{\varepsilon}(t+\tau) = \overline{\varepsilon}(t) + 2\tau \sum_{\overline{\omega}_h} \hbar r y_{\check{t}} f_1 + 2\tau \left[R_1 p_1 y_{\check{t}}|_{R_1} - R_0 p_0 y_{\check{t}}|_{R_0} \right]. \tag{6.3}$$

The condition of nonnegativeness of the discrete analog of the energy integral gives the stability condition of the scheme (4.6) through (4.12).

Let us estimate the quality $\overline{\varepsilon}(t)$, assuming its nonnegativeness for any value of its arguments. Summing (6.3) in t from τ to a certain $t_1 (\tau < t_1 \leq T)$, we obtain

$$\overline{\varepsilon}(t+\tau) = \overline{\varepsilon}(\tau) + \sum_{t'=\tau}^{t_1} 2\tau \sum_{\overline{\omega}_h} \hbar r y_{\check{t}} f_1 + \sum_{t'=\tau} 2\tau \left[R_1 p_1 y_{\check{t}}|_{R_1} - R_0 p_0 y_{\check{t}}|_{R_0} \right]. \tag{6.4}$$

The second term on the right-hand side of (6.4) can be estimated with the ε-inequality, whereas for the last term we have[18] that

$$\sum_{t'=\tau}^{t_1} 2\tau [R_1 p_1 v_N - R_0 p_0 v_0] = \sum_{t'=\tau}^{t_1} 2\tau \left\{ R_1 \left[(p_1 y(R_1,t))_{\check{t}} - \frac{1}{2} (\hat{y}_N (p_1)_t + \check{y}_N (p_1)_{\bar{t}}) \right] \right.$$
$$\left. - R_0 \left[(p_0 y(R_0,t))_{\check{t}} - \frac{1}{2} (\hat{y}_0 (p_0)_t + \check{y}_0 (p_0)_{\bar{t}}) \right] \right\} = I_1 - I_2 + I_3,$$

where

$$I_1 = R_1 [p_1(t_1+\tau) y(R_1, t_1+\tau) + p_1(t_1) y(R_1, t_1)]$$
$$- R_0 [p_0(t_1+\tau) y(R_0, t_1+\tau) + p_0(t_1) y(R_0, t_1)],$$
$$I_2 = R_1 [p_1(\tau) y(R_1,\tau) + p_1(0) y(R_1,0)] - R_0 [p_0(\tau) y(R_0,\tau) + p_0(0) y(R_0,0)],$$
$$I_3 = \sum_{t'=\tau}^{t_1} 2\tau \left[R_0 \frac{1}{2} (\hat{y}_0 (p_0)_t + \check{y}_0 (p_0)_{\bar{t}}) - R_1 \frac{1}{2} (\hat{y}_N (p_1)_t + \check{y}_N (p_1)_{\bar{t}}) \right].$$

Then, application of the ε-inequality to the term I_1 gives

$$|I_1| \leq \frac{R_1}{4\varepsilon_2} \max_{t=t_1,t_1+\tau} |p_1(t)|^2 + \frac{R_0}{4\varepsilon_3} \max_{t=t_1,t_1+\tau} |p_0(t)|^2 + R_1 \varepsilon_2 |\hat{y}_N + y_N|^2 + R_0 \varepsilon_3 |\hat{y}_0 + y_0|^2.$$

And since

$$|\hat{y}_N + y_N|^2 + |\hat{y}_0 + y_0|^2 \le M_1 \sum_{\omega_h^+} h\bar{r}\frac{1}{4}\left(\bar{\varepsilon}_r + \check{\bar{\varepsilon}}_r\right)^2\Big|_{t=t_1+\tau} \le M_2\overline{\mathcal{E}}(t_1+\tau),$$

we finally have

$$|I_1| \le M\left\{\max_{t=t_1,t_1+\tau}|p_1(t)|^2 + \max_{t=t_1,t_1+\tau}|p_0(t)|^2 + \overline{\mathcal{E}}(t_1+\tau)\right\}.$$

In a similar manner, we conclude that

$$|I_2| \le M\left\{\max_{t=0,\tau}|p_1(t)|^2 + \max_{t=0,\tau}|p_0(t)|^2 + \overline{\mathcal{E}}(\tau)\right\},$$

and

$$|I_3| \le \varepsilon_1 R_0 \sum_{t'=\tau}^{t_1}\tau \max_{t'-\tau,t'+\tau}|y_0|^2 + \frac{R_0}{4\varepsilon_1}\sum_{t'=\tau}^{t_1}\tau\left[((p_0)_t)^2 + ((p_0)_{\bar{t}})^2\right]$$

$$+\varepsilon_2 R_1 \sum_{t'=\tau}^{t_1}\tau \max_{t'-\tau,t'+\tau}|y_N|^2 + \frac{R_1}{4\varepsilon_2}\sum_{t'=\tau}^{t_1}\tau\left[((p_1)_t)^2 + ((p_1)_{\bar{t}})^2\right].$$

Taking into consideration that

$$\sum_{t'-\tau}^{t_1}\tau \max_{t'-\tau,t'+\tau}\left(|y_0|^2 + |y_N|^2\right) \le M_3\sum_{t'=0}^{T}\tau\overline{\mathcal{E}}(t'),$$

and applying the discrete Gronwall inequality [5], we obtain

$$\overline{\mathcal{E}}(t_1+\tau) \le M_4\overline{\mathcal{E}}(\tau) + M_5\left[\max_{t=0,\tau,t_1,t_1+\tau}\left(|p_0(t)|^2 + |p_1(t)|^2\right)\right]$$

$$+M_6\sum_{t'=\tau}^{t_1}\tau\left[((p_0)_{\bar{t}})^2 + ((p_1)_{\bar{t}})^2\right] + M_7\sum_{t'=\tau}^{t_1}\tau\sum_{\overline{\omega}_h}\hbar r f_1^2. \tag{6.5}$$

The main difficulty in obtaining an estimate for the discrete analog of the energy integral $\overline{\mathcal{E}}$ at the moment of discrete time $t=\tau$ is the term $\sum_{\omega_h^+} h\bar{r}\varepsilon_{11}\overline{E}_r^2(0)$. To estimate it, let us multiply scalarly the discrete analog of the Maxwell equation (4.7) at $t=0$ by the discrete approximation of potential μ:

$$((\bar{r}\overline{D}_r)_r, \mu) = (rf_2, \mu).$$

Taking into consideration approximations for the state equations(4.8) and applying the Cauchy-Schwarz inequality (as well as the ε-inequality), it is easy to show that

$$\sum_{\omega_h^+}\bar{r}h\overline{E}_r^2(0) \le M\left\{\sum_{\omega_h^+}h\bar{r}\overline{\lambda}^2(0) + \sum_{\omega_h^+}h\bar{r}\bar{\varepsilon}_r^2(0) + \sum_{\omega_h^+}h\bar{r}\bar{\varepsilon}_\theta^2(0)\right\}, \tag{6.6}$$

where $rf_2 = (\bar{r}\lambda)_r$, $\bar{\lambda}^{(+1)} = \bar{D}_r^{(+1)}$ for $r = R_0$. Then, using the definition(6.2), inequality (6.6), and the relationship (3.6) with the inequality (6.5), we obtain the a priori estimate[19]

$$\begin{aligned}\bar{\mathcal{E}}(t_1+\tau) \le M \Bigg\{ & \rho \sum_{\bar{\omega}_h} \hbar r y_t^2(0) + \sum_{\omega_h^+} h\bar{r} \Bigg\{ c_{11} \left[\bar{\varepsilon}_r^2(0) + \left(\frac{\tau}{2} (\bar{\varepsilon}_r)_t (0) \right)^2 \right] \\ & + 2c_{12} \left[\bar{\varepsilon}_r(0)\bar{\varepsilon}_\theta(0) + \frac{\tau}{2} \left(\bar{\varepsilon}_r(0)(\bar{\varepsilon}_\theta)_t(0) + \bar{\varepsilon}_\theta(0)(\bar{\varepsilon}_r)_t(0) \right) \right] \\ & + c_{22} \left[\bar{\varepsilon}_\theta^2(0) + \left(\frac{\tau}{2} (\bar{\varepsilon}_\theta)_t(0) \right)^2 \right] \Bigg\} + \max_{t=0,\tau,t_1,t_1+\tau} \left(|p_0(t)|^2 + |p_1(t)|^2 \right) \\ & + \sum_{t'=\tau}^{t_1} \tau \left[((p_0)_{\bar{t}})^2 + ((p_1)_{\bar{t}})^2 \right] + \sum_{\omega_h^+} h\bar{r}\bar{\lambda}^2(0) + \sum_{t'=\tau}^{t_1} \tau \sum_{\bar{\omega}_h} r\hbar f_1^2 \Bigg\}. \end{aligned} \tag{6.7}$$

Let us note that the assumption $V(t) = 0$ adopted here does not restrict the generality of the problem. In fact, by introducing a new unknown function $\phi_1 = \phi - L(r,t)$,[20] we can easily reduce the original differential problem (3.1) through (3.5) to the problem with homogeneous boundary conditions for the potential. The explicit form of the function $L(r,t)$ is

$$L(r,t) = \left[\frac{2r}{R_0 - R_1} + \frac{R_1 + R_0}{R_1 - R_0} \right] V(t).$$

Hence, for the problem with homogeneous boundary conditions for potential, we should only change the right-hand part of the equation of motion and boundary conditions for stress:

$$f_1'(r,t) = f_1(r,t) + \frac{2e_{12}V(t)}{r(R_1 - R_0)}, \quad p_i' = p_i + \frac{2e_{11}V(t)}{R_1 - R_0}, \quad i = 0,1.$$

Estimation of the discrete analog of the energy integral of the electromechanical system (6.7) has been obtained under the conditions of its nonnegativeness. Let us find this condition in an explicit form. Without attracting "deformational" terms to prove nonnegativeness of the seminorm

$$\sum_{\omega_h^+} h\bar{r} \left[\varepsilon_{11} \frac{1}{4} \left(\bar{E}_r + \check{\bar{E}}_r \right)^2 - \frac{\tau^2}{4} \left((\bar{E}_r)_{\bar{t}} \right)^2 \right] (\tau)$$

for the electric field seems to be impossible. Thus, we derive an estimate

$$\frac{\tau^2}{4} \sum_{\omega_h^+} h\bar{r} \left((\bar{E}_r)_{\bar{t}} \right)^2$$

as follows. From the discrete approximation of the Maxwell equation[21]

$$\sum_{\omega_h^+} h\bar{r}\bar{E}_r\bar{D}_r = \sum_{\omega_h^+} h\bar{r}\bar{\lambda}\bar{E}_r,$$

we get that

$$\varepsilon_{11}\sum_{\omega_h^+} h\bar{r}((\overline{E}_r)_{\bar{t}})^2 \le \frac{2}{\varepsilon_{11}}\left\{e_{11}^2\sum_{\omega_h^+} h\bar{r}((\bar{\varepsilon}_r)_{\bar{t}})^2 + e_{12}^2((\bar{\varepsilon}_r)_{\bar{t}})^2\right\}.$$

Here, we take into consideration that by virtue of the choice $\overline{\lambda}$, the Maxwell equation, and the approximation $(\overline{D}_r)_{\bar{t}} = 0$, we have that $\overline{\lambda}_{\bar{t}} = 0$. As a result, we get the inequality

$$\begin{aligned}\overline{\mathcal{E}}(t) \ge \rho\sum_{\overline{\omega}_h}\hbar r y_{\bar{t}}^2 &+ \sum_{\omega_h^+} h\bar{r}\left\{\left[c_{11}\frac{1}{4}(\bar{\varepsilon}_r+\check{\bar{\varepsilon}}_r)^2 - \frac{\tau^2}{4}\left(c_{11}+\frac{2e_{11}^2}{\varepsilon_{11}}\right)((\bar{\varepsilon}_r)_{\bar{t}})^2\right]\right.\\ &+ c_{12}\left[\bar{\varepsilon}_r\bar{\varepsilon}_\theta + \check{\bar{\varepsilon}}_r\check{\bar{\varepsilon}}_\theta - \tau^2(\bar{\varepsilon}_r)_{\bar{t}}(\bar{\varepsilon}_\theta)_{\bar{t}}\right]\\ &+\left.\left[c_{22}\frac{1}{4}(\bar{\varepsilon}_\theta+\check{\bar{\varepsilon}}_\theta)^2 - \frac{\tau^2}{4}\left(c_{22}+\frac{2e_{12}^2}{\varepsilon_{11}}\right)(\bar{\varepsilon}_\theta)_{\bar{t}}^2\right]\right\} + \frac{\varepsilon_{11}}{4}\sum_{\omega_h^+} h\bar{r}\left(\overline{E}_r+\check{\overline{E}}_r\right)^2.\end{aligned}$$

With regard to (3.6), it is clear that the nonnegativeness of $\overline{\mathcal{E}}(t)$ will be provided if we prove the inequality

$$\begin{aligned}\rho\sum_{\overline{\omega}_h}\hbar r y_{\bar{t}}^2 - \tau^2&\left\{\left[\frac{c_{11}}{4}+\frac{1}{2}\frac{e_{11}^2}{\varepsilon_{11}}\right]\sum_{\omega_h^+} h\bar{r}((\bar{\varepsilon}_r)_{\bar{t}})^2\right.\\ &+\left.\left[\frac{c_{22}}{4}+\frac{1}{2}\frac{e_{12}}{\varepsilon_{11}}\right]\sum_{\omega_h^+} h\bar{r}((\bar{\varepsilon}_\theta)_{\bar{t}})^2 + \frac{c_{12}}{2}(\bar{\varepsilon}_r)_{\bar{t}}(\bar{\varepsilon}_\theta)_{\bar{t}}\right\} \ge 0. \qquad (6.8)\end{aligned}$$

Taking into consideration the inequalities

$$\begin{aligned}\sum_{\omega_h^+} h\bar{r}(\bar{\varepsilon}_r)^2 &= \sum_{\omega_h^+} h\bar{r}(y_{\bar{r}})^2 \le \frac{2}{h^2}\left[\sum_{\omega_h^+} h\bar{r}y^2 + \sum_{\omega_h^+} h\bar{r}\left(y^{(-1)}\right)^2\right]\\ &= \frac{2}{h^2}\left[\sum_{\omega_h^+} hr\frac{\bar{r}}{r}y^2 + \sum_{\omega_h^-} hr\frac{\bar{r}^{(+1)}}{r}y^2\right] \le \frac{4}{h^2}\left(1+\frac{h}{2R_0}\right)\sum_{\overline{\omega}_h}\hbar r y^2,\\ \sum_{\omega_h^+} h\bar{r}(\bar{\varepsilon}_\theta)^2 &= \sum_{\omega_h^+} h\bar{r}\left(\frac{y+y^{(-1)}}{2\bar{r}}\right)^2 \le \frac{1}{2}\left[\sum_{\omega_h^+} h\frac{y^2}{\bar{r}} + \sum_{\omega_h^+} h\frac{\left(y^{(-1)}\right)^2}{\bar{r}}\right]\\ &= \frac{1}{2}\left[\sum_{\omega_h^+} hr\frac{1}{\bar{r}r}y^2 + \sum_{\omega_h^-} hr\frac{1}{\bar{r}^{(+1)}r}y^2\right] \le \frac{1}{R_0^2}\sum_{\overline{\omega}_h}\hbar r y^2,\end{aligned}$$

and

$$\sum_{\omega_h^+}\bar{r}h\bar{\varepsilon}_r\bar{\varepsilon}_\theta = \frac{1}{h}\sum_{\omega_h^+} hr\frac{y^2-\left(y^{(-1)}\right)^2}{2\bar{r}} \le \frac{1}{2R_0h}\sum_{\overline{\omega}_h}\hbar r y^2,$$

we conclude that (6.8) is satisfied if the inequality

$$\rho - \tau^2 \left[\frac{4}{h^2}\left(1+\frac{h}{2R_0}\right)\left(\frac{c_{11}}{4}+\frac{e_{11}^2}{2\varepsilon_{11}}\right) + \frac{1}{R_0^2}\left(\frac{c_{22}}{4}+\frac{e_{12}^2}{2\varepsilon_{11}}\right) + \frac{c_{12}}{4R_0 h}\right] \geq \varepsilon, \quad \text{where } \varepsilon > 0$$

holds. If we note that $\delta = e_{11}^2/(\varepsilon_{11}c_{11})$ is the coupling coefficient of the electromechanical system[22] described by the model (3.1) through (3.5), and $c = \sqrt{c_{11}(1+\delta)/\rho}$ is the velocity of propagation of mixed electroelastic waves that are the solution of the coupled dynamic problem of electroelasticity, then the stability condition for the discrete scheme (4.6) through (4.12) can be finally written in the form

$$\tau \leq \frac{h}{c}\left\{\left(1-\frac{\varepsilon}{\rho}\right) \Big/ \left[\left(1+\frac{h}{2R_0}\right)\left(1+\frac{\delta}{1+\delta}\right) + \frac{c_{12}}{4R_0 c_{11}(1+\delta)}h \right.\right.$$
$$\left.\left. + \frac{h^2}{4R_0^2 c_{11}(1+\delta)}\left(c_{22}+\frac{2e_{12}^2}{\varepsilon_{11}}\right)\right]\right\}^{\frac{1}{2}}. \tag{6.9}$$

This completes the proof of the following theorem:

THEOREM 6.1. *Under the stability conditions (6.9), the solution of the discrete problem (4.6) through (4.12) satisfies the energy estimate (6.7), where the discrete analog of energy integral is defined by (6.2).*

Remark 6.1. The stability condition (6.9) for the zero-coupling coefficient[23] coincides in the dominant part with the stability condition for a discrete scheme obtained for a noncoupled problem of electroelasticity (Moskalov [15]).

Remark 6.2. Of course, the stability condition (6.9) has a quite definite physical meaning. In the case of circular preliminary polarization, when mechanical and electric fields appear to be uncoupled, the velocity of propagation of pure elastic waves $c_0 = \sqrt{c_{11}/\rho}$ defines the stability of discrete schemes [15]. Whereas, in the case of radial preliminary polarization, the coupling effect manifests itself, and time-step discretization depends on the velocity of propagation of mixed electroelastic waves $c = \sqrt{c_{11}(1+\delta)/\rho}$. Hence, the derived condition gives a Courant-Fideriches-Lewy-type stability condition for the case of coupled dynamic electroelasticity.

7. CONVERGENCE OF THE SCHEME ON THE CLASS OF GENERALIZED SOLUTIONS FROM $W_2^4(Q_T)$

The error of the scheme (4.6) through (4.12),

$$z = y - u, \quad \zeta = \mu - \phi,$$

is the solution of the operator difference scheme

$$\begin{cases} D_1 z_{\bar{t}t} + A_1 z + C_1 \zeta = \psi & t \in \omega_\tau, \\ A_2 \zeta + C_2 z = \kappa, & t \in \overline{\omega}_\tau, \\ z = 0, \quad D_1 z_t = \psi, & t = 0. \end{cases} \tag{7.1}$$

The functions $z = z(t)$, $\zeta = \zeta(t)$ for each $t \in \overline{\omega}_\tau$ are elements of the Hilbert spaces defined in Section 4, whereas the operators of the scheme are defined as

$$A_1 z = \begin{cases} -\frac{2}{h}\bar{r}^{(+1)}\breve{\sigma}_r^{(+1)} + \breve{\sigma}_\theta^{(+1)}, & r = R_0, \\ -\left(\bar{r}\breve{\sigma}_r\right)_r + \frac{\breve{\sigma}_\theta^{(+1)} + \breve{\sigma}_\theta}{2}, & R_0 < r < R_1, \\ \frac{2}{h}\bar{r}\breve{\sigma}_r + \breve{\sigma}_\theta, & r = R_1, \end{cases}$$

$$C_1 \zeta = \begin{cases} \frac{2e_{11}}{h}\bar{r}^{(+1)}\breve{E}_r^{(+1)} - e_{12}\breve{E}_r^{(+1)}, & r = R_0, \\ e_{11}\left(\bar{r}\breve{E}_r\right)_r - e_{12}\frac{\breve{E}_r^{(+1)} + \breve{E}_r}{2}, & R_0 < r < R_1, \\ -\frac{2e_{11}}{h}\bar{r}\breve{E}_r + e_{12}\breve{E}_r, & r = R_1, \end{cases}$$

$$D_1 z = r\rho z, \; A_2\zeta = \varepsilon_{11}\left(\bar{r}\breve{E}_r\right)_r, \; C_2 z = \left[\bar{r}\left(e_{12}\breve{\varepsilon}_\theta + e_{11}\breve{\varepsilon}_r\right)\right]_r,$$

where

$$\breve{\sigma}_r = c_{11}\breve{\varepsilon}_r + c_{12}\breve{\varepsilon}_\theta, \; \breve{\sigma}_\theta = c_{12}\breve{\varepsilon}_r + c_{22}\breve{\varepsilon}_\theta, \; \breve{E}_r = -\zeta_{\bar{r}}, \; \breve{\varepsilon}_r = z_{\bar{r}}, \; \breve{\varepsilon}_\theta = \frac{z + z^{(-1)}}{2\bar{r}}.$$

The approximation error of the scheme (4.6) through (4.12) is defined as

$$\psi = -\rho r u_{\bar{t}t} + \overline{\psi} \text{ for } t \in \omega_\tau, \text{ and } \psi = \rho r u_1 - \rho r u_t + \frac{\tau}{2}\psi \text{ for } t = 0,$$

$$\kappa = -\left(\bar{r}\overset{*}{D}_r\right)_r = -\left(\bar{r}\left(\varepsilon_{11}\overset{*}{E}_r + e_{12}\overset{*}{\varepsilon}_\theta + e_{11}\overset{*}{\varepsilon}_r\right)\right)_r,$$

where

$$\overline{\psi} = \begin{cases} \left(\bar{r}\overset{*}{\sigma}_r\right)_r - \frac{\overset{*}{\sigma}_\theta^{(+1)} + \overset{*}{\sigma}_\theta}{2} + rf_1 & \text{for } r \in \omega_h, \\ \frac{2}{h}\bar{r}^{(+1)}\overset{*}{\sigma}_r^{(+1)} - \overset{*}{\sigma}_\theta^{(+1)} + rf_1 - \frac{2}{h}rp_0, & \text{for } r = R_0, \\ -\frac{2}{h}\bar{r}\overset{*}{\sigma}_r - \overset{*}{\sigma}_\theta + rf_1 + \frac{2}{h}rp_1, & \text{for } r = R_1, \end{cases}$$

$$\overset{*}{\sigma}_r = c_{11}\overset{*}{\varepsilon}_r + c_{12}\overset{*}{\varepsilon}_\theta - e_{11}\overset{*}{E}_r, \quad \overset{*}{\sigma}_\theta = c_{12}\overset{*}{\varepsilon}_r + c_{22}\overset{*}{\varepsilon}_\theta - e_{12}\overset{*}{E}_r,$$

$$\overset{*}{E}_r = -\phi_{\bar{r}}, \; \overset{*}{\varepsilon}_r = u_{\bar{r}}, \quad \overset{*}{\varepsilon}_\theta = \frac{u + u^{(-1)}}{2\bar{r}}.$$

If the sought-for solution of the problem (3.1) through (3.5) belongs to the Sobolev space $W_2^4(Q_T)$, the use of Taylor's expansion with the integral form of the remainder leads to the conclusion that the error of approximation for and $t \in \overline{\omega}_\tau$ can be presented in the form

$$\psi = \breve{\psi} + \delta(h)\overset{*}{\psi},$$

where

$$\delta(h) = \begin{cases} 0, & \text{for } r \in \omega_h, \\ 2/h, & \text{for } r = R_0, \\ -2/h, & \text{for } r = R_1, \end{cases}$$

and that the functionals $\breve{\psi}, \overset{*}{\psi}, \kappa$ have the second order of smallness in grid steps; that is,

$$\breve{\psi} = O(h^2 + \tau^2),\ \overset{*}{\psi} = O(h^2 + \tau^2),\ \kappa = O(h^2).$$

Now, to obtain an accuracy estimate for the discrete scheme (4.6) through (4.12), we use the a priori estimate (6.7) established under the condition (6.9). After transformations, we have the accuracy estimate

$$\begin{aligned} \breve{\mathcal{E}}(t_1+\tau) \leq M \Bigg\{ & \sum_{\omega_h} hr\breve{\psi}(r,0) + \sum_{\omega_h^+} h\bar{r}\left[c_{11}\breve{\varepsilon}_r^2(0) + 2\breve{\varepsilon}_r(0)\breve{\varepsilon}_\theta(0) + c_{22}\breve{\varepsilon}_\theta^2(0)\right] \\ & + \max_{t=0,\tau,t_1,t_1+\tau}\left(\left|\overset{*}{\psi}(R_0,t)\right|^2 + \left|\overset{*}{\psi}(R_1,t)\right|^2\right) + \sum_{t'=\tau}^{t_1}\tau\left[\left(\overset{*}{\psi}_{\bar{t}}(R_0,t')\right)^2 + \left(\overset{*}{\psi}_{\bar{t}}(R_1,t')\right)^2\right] \\ & + \sum_{t'=\tau}^{t_1}\tau\sum_{\omega_h} hr\breve{\psi}^2(r,t') + \sum_{\omega_h^+} h\bar{r}\kappa_1^2(0) \Bigg\}, \end{aligned} \tag{7.2}$$

where $\breve{\mathcal{E}}(t_1+\tau)$ is obtained by the replacement in $\overline{\mathcal{E}}(t_1+\tau)$ of all y by z and ϕ by ζ; $r\kappa = (\bar{r}\kappa_1)_r$, $\kappa_1^{(+1)} = \overset{*}{D}{}_r^{(+1)}$ when $r = R_0$, and the functions $\breve{\varepsilon}_r, \breve{\varepsilon}_\theta, \kappa_1$ on the right-hand side of (7.2) are computed for $t = 0$. Of course, when the sought-for solution is from the class $W_2^4(Q_T)$, both quantities $\overset{*}{\psi}_{\bar{t}}$ and κ_1 also have the second order of smallness with respect to grid steps. Hence, we come to the main result of this section.

Theorem 7.1. *If the stability condition (6.9) holds, the solution of the discrete problem (4.6) through (4.12) converges to the sought-for solution of the original problem (3.1) through (3.5) from the class $W_2^4(Q_T)$ with the second order with respect to space-time grid steps. Such solutions satisfy the accuracy estimate (7.2).*

Remark 7.1. If the equation of motion and the Maxwell equation are coupled between themselves only by the state equations but are not coupled by the boundary conditions for stresses,[24] then the problem is essentially simplified. In such a case, a stronger result than that stated in Theorem 7.1 can be obtained. Namely, if the generalized solution of the original problem is assumed to be from the class $W_2^2(Q_T)$, then the second order of convergence can be preserved for the discrete scheme in a weaker than L^2 metric. Results of this type for a single wave equation were discussed earlier in Djuraev and Moskalov [4].

8. NUMERICAL RESULTS AND CONCLUSIONS

In this section, we present some numerical results on the process of coupled electroelastic oscillations of a hollow piezoceramic cylinder under strong and weak levels of couplings. The model (3.1) through (3.5) is appropriate to investigate this process when the thickness-to-length ratio is small. We assume the unexcited state of piezoceramics at the initial moment of time.[25] As for boundary conditions, we assume that there are no stresses on the exterior and interior surfaces, whereas a given potential difference $2V$ is maintained between them.

Analyzing the dynamics of displacements in time on the external surface of cylinders represents the main interest of our computational experiment. Previously, a comparison was made between the weakly coupled case (circular polarization) and the strongly coupled case (radial polarization) [13]. Numerical results showed increases in the amplitude of oscillations in the latter case. This led us to consider the radial preliminary polarization case in this article. Furthermore, increases of oscillations manifest essentially for thin cylinders.

Due to an inconsistency between the initial and boundary conditions at the initial moment of time, stresses are defined by a discontinuous function. However, since stresses reflect the contraction-extension process of cylinders, their computation is an important part of the numerical experiment. Figures 1 and 2 show stresses and displacements in hollow, thin cylinders of different radii.

Since our computational models were derived with nonstationary problems in mind, we analyze the change of displacements in time on the external surface of the cylinders. This process characterizes cylinders as piezoceramic vibrators. Two examples are shown in Figures 3 and 4. In the general nonstationary case, accounting for coupling effects between electric and elastic fields in anisotropic materials may essentially influence the results of computations. Steep gradients in computed functions require appropriate mathematical tools to deal with such phenomena. The concept of generalized solutions provides a unified framework for the derivation of such computational models and their justification.

The model we have considered is typical in coupled field theory, where the interinfluence of physical fields of different nature is essential to obtain a plausible picture of the underlying phenomena. Usually, at least one of the equations in the model is a partial differential equation of hyperbolic type. A connection of such a hyperbolic-type operator with elliptic or parabolic modes of the model leads to a situation where numerical methods become natural and the most efficient way of solving problems arising from coupled field theory. Mathematical modeling in this area requires approaches that can be applied even if a solution of the problem does not possess the excessive smooth-

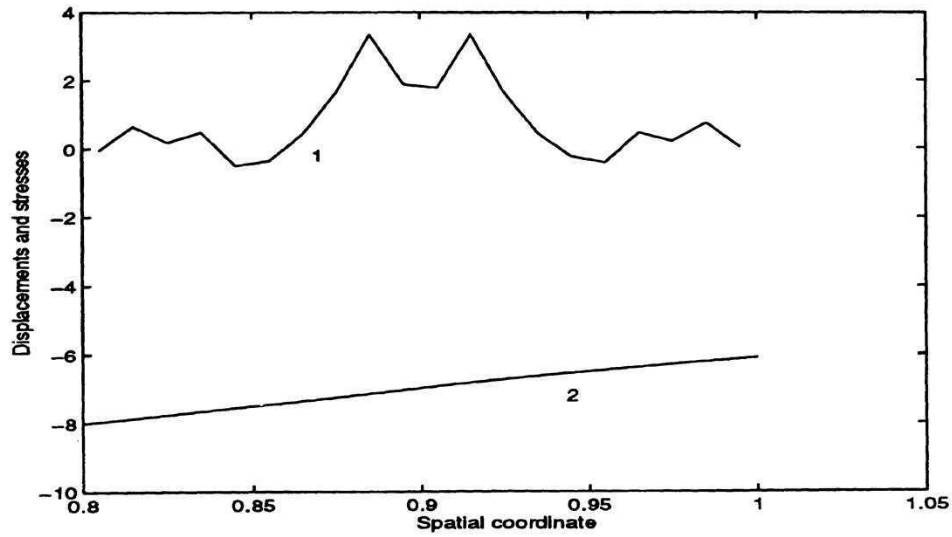

Fig. 1. Radial stresses (curve 1) and displacements (curve 2) at t = 10 (radial preliminary polarization; thickness 1 = 0.2).

ness often imposed as an a priori assumption. Mathematical challenges and the practical importance of these problems stimulate interest from mathematicians, engineers, and scientist.

Of course, the approach based on the coupling procedures is a natural way to reflect additional information about the system and to implement it into mathematical models. However, in mathematical modeling and computational experiments, we always use the implicit assumption that the problem can effectively be reduced to a finite set of equations, inequalities, or inclusions. Although such an approach will remain a powerful tool for investigation of the real-world problems in the foreseeable future, it is very important to pay attention to models of varying complexity. Fixing a degree of coupling in mathematical models implies a thorough investigation of system stability at the specified coupling level.

All real processes, dynamic systems, and phenomena describe a transformation of different types of energy, implying that, in general, mathematical models applied to them should have integral rather than differential features. Clearly, for example, a border between two different media might not be described appropriately by any differential equation due to a jump of physical parameters. A similar situation arises when we try to describe a nonhomogeneous medium. Probably one of the most demonstrative examples of difficulties involved in the mathematical modeling of such media is provided by nonlocal type models. Along with classical applications of such models for the description of macrosystems and microstructures,[26] nonlocal type models are typical when we address physical problems of mathematical modeling by using an extended thermodynamic approach (Jou, Casa-Vasquez, and Lebon [6]; Muller and Ruggeri [16]). The hyperbolic

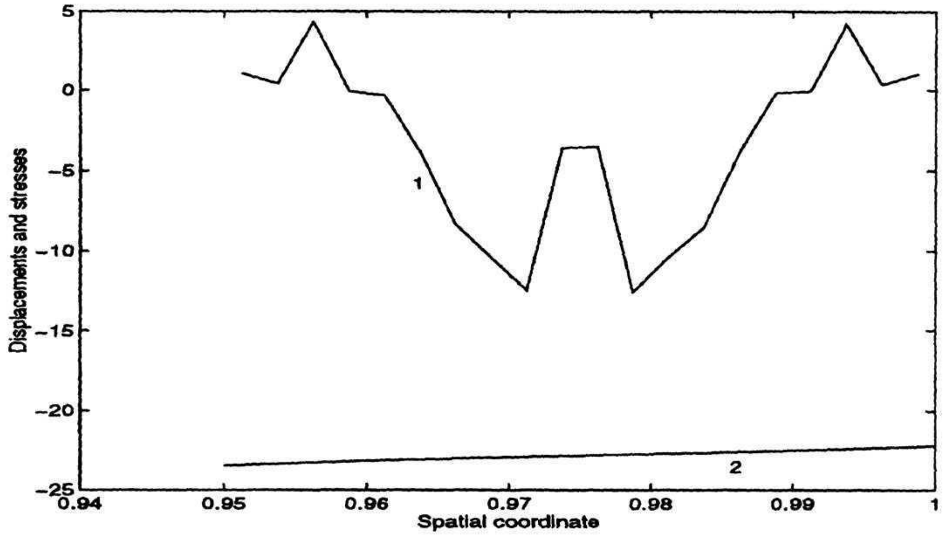

Fig. 2. Radial stresses (curve 1) and displacements (curve 2) at $t = 10$ (radial preliminary polarization; thickness 1 = 0.05).

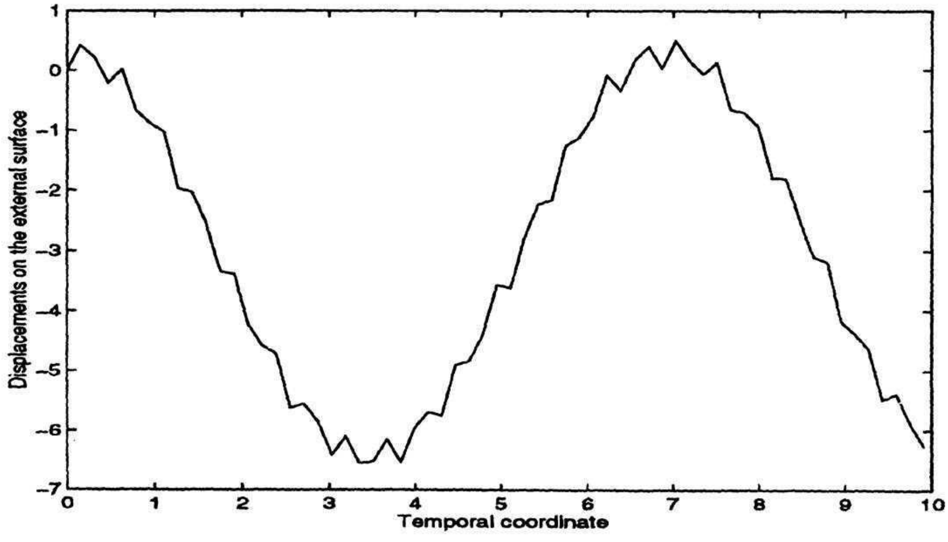

Fig. 3. Time dependency of displacements on the external surface of the cylinder (radial preliminary polarization; thickness 1 = 0.2).

nature of arising models provides an important area of further investigations. In general, many problems in coupled field theory[27] may not have an adequate description in mathematical models if a priori assumptions of excessive[28] smoothness areimposed on their solutions.

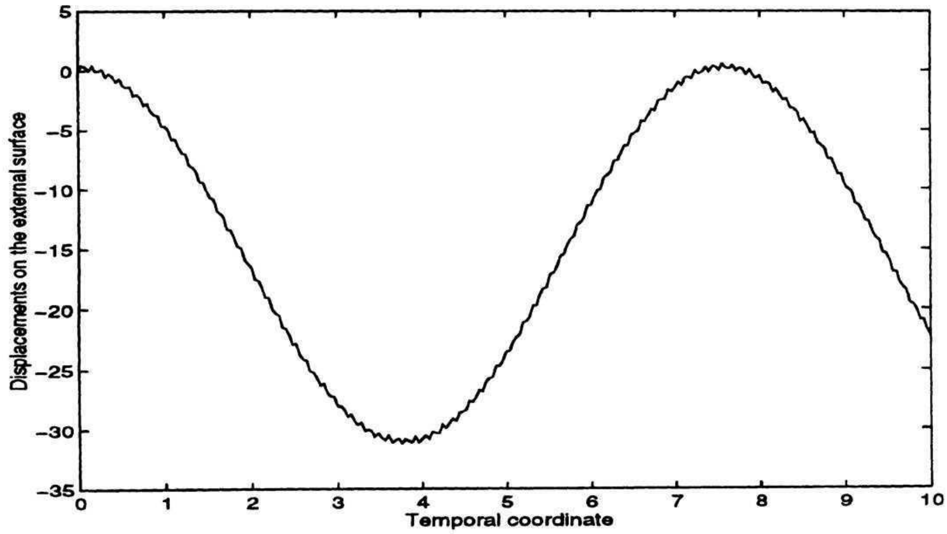

Fig. 4. Time dependency of displacements on the external surface of the cylinder (radial preliminary polarization; thickness 1 = 0.05).

A connection between variational principles and computational models for new areas of applications provides many challenging problems (Yu [21]). On the other hand, linear models in coupled field theory give important guidelines for further development of theory in the nonlinear case. Recently, mixed[29] variational principles were proposed in nonlinear electroelasticity [20]. Accounting for discontinuities and steep gradients in the solutions, it is important to be able to achieve a trade-off between a priori assumptions on solution smoothness and computational efficiency of the numerical procedures derived from physical principles.

As an ultimate goal, we would like to generalize the presented technique to nonlocal models arising from micro- and macro-levels of description of real dynamic systems. Some results in this field have been already published in Melnik [10].

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NOTES

1. A priori imposed on the solution.
2. Usually for mechanical components of electroelastic fields.
3. It is assumed that electrodes cover the surface of the cylinder, and the active electric load is given by the potential difference $2V$.

4. Analogous notations are used with respect to the spacial variable r with the spacial step of discretization h.

5. In the acoustic range of frequencies, it is the forced electroelastic equation of dielectrics.

6. It is easy to notice that the second derivative of φ with respect to r is present in the first equation, which is primarily responsible for the elastic field; also the second derivative of u with respect to r is present in the second equation, which is primarily responsible for the electric field.

7. In particular, such a situation holds for the range of acoustic frequencies [1].

8. Note also that conservative properties for such discrete models follow from the Noether theorem [18].

9. This is a Hilbert space that consists of elements $u(r,t) \in L^2(Q_T)$, which have square summable generalized derivatives $\partial u/\partial r$.

10. This equality is a consequence of the state equations (3.4).

11. Using (3.8) with $\zeta = \phi$, and $t = 0, t_1$.

12. In what follows, we denote functions of discrete variable $r \in \overline{\omega}_h$ and continuous $t \in \overline{I}$ by an upper tilde.

13. Using grid formulas for summation by parts.

14. Since boundary conditions for potential are main, they do not follow directly from (3.7).

15. It can be assumed in the acoustic range of frequencies, for example, that the piezoelectric oscillations are accompanied only by negligible magnetic effects.

16. We also need two obvious results: $\frac{\partial u}{\partial t}v = \frac{\partial}{\partial t}(uv) - u\frac{\partial v}{\partial t}$ and $|(u,v)| \leq ||u||||v|| \leq \varepsilon||u||^2 + \frac{1}{4\varepsilon}||v||^2 \ \forall\varepsilon > 0$; the first one is a consequence of the Newton-Libniz formula, whereas the second is known as the ε-inequality.

17. Because of (5.1).

18. Here, we use $u_{\bar{t}}v = (uv)_{\bar{t}} - (\hat{u}v_t + \check{u}v_{\bar{t}})/2$.

19. We also use the inequality

$$\frac{1}{4}(\overline{E}_r + \check{\overline{E}}_r)^2(\tau) - \frac{\tau^2}{4}((\overline{E}_r)_{\bar{t}})^2(\tau) \leq \overline{E}_r^2(0) + \frac{\tau^2}{4}((\overline{E}_r)_{\bar{t}})^2(0)$$

and equality

$$[\overline{\varepsilon}_r\overline{\varepsilon}_\theta + \check{\overline{\varepsilon}}_r\check{\overline{\varepsilon}}_\theta - \tau^2(\overline{\varepsilon}_r)_{\bar{t}}(\overline{\varepsilon}_\theta)_{\bar{t}}]|_{t=\tau} = 2\overline{\varepsilon}_r(0)\overline{\varepsilon}_\theta(0) + \tau[\overline{\varepsilon}_r(0)(\overline{\varepsilon}_\theta)_t(0) + \overline{\varepsilon}_\theta(0)(\overline{\varepsilon}_r)_t(0)].$$

20. Here, $L(r,t)$ is a linear function in r that equals $V(t)$ for $r = R_0$ and $-V(t)$ for $r = R_1$.

21. And the obvious inequality $(a+b)^2 \leq 2(a^2+b^2)$.

22. It characterizes the effect of power transformation in piezoelectric material better than the set of elastic, dielectric, and piezoelectric constants [1].

23. That is, $\delta = 0$ when coupling between electric and elastic fields is negligible.

24. For example, displacements are given on the boundary.

25. We use the PZT-4 cylinder, which after a scaling procedure is characterized by the coefficients $2V = 1$, $\rho = 1$, $R_1 = 1$, $c_{11} = 0.82734$, $c_{12} = 0.53453$, $c_{22} = 1$, $e_{11} = 0.54027$, $e_{12} = -0.18605$, and $\varepsilon_{11} = 1$.

26. As in climate modeling and semiconductor device simulation.

27. Arising from studying microstructures as well as macrosystems.

28. With respect to real solutions.

29. In a sense that all main variables such as stresses, electric field, displacements, and electric potential are considered as field variables.

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