Complete Solutions to a Class of Nonconvex Variational/Boundary value Problems with Implications for Computational Science

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ABSTRACT:
The fundamental difficulty in variational analysis, boundary value problems, numerical computation, and global optimization is mainly due to nonconvexity. In static systems, the nonconvexity of problems usually leads to multi-solutions in the related nonlinear differential equations. How to identify the extremality of these critical solutions is a challenging task not only in nonlinear analysis, but also in numerical methods and computational science. In nonlinear dynamics, the nonconvexity usually leads to the so-called chaotic phenomena. Many nonconvex problems in global optimization are NP-hard. In nonlinear elasticity, even some qualitative questions such as the existence, uniqueness, regularity and stability of solutions have been listed as the outstanding open problems by Sir John Ball.

In this talk, the speaker will present a set of complete analytic solutions to certain nonconvex variational/boundary value problems whose total potential is either quadratic-exponential functional or high order (at least 4th order) polynomials. The speaker will first show that by using the canonical duality theory by the speaker, these partial differential equations can be converted into certain algebraic (tensor) equations, which can, in principle, be solved to obtain a complete set of analytic solutions. Both global minimizer and local extrema are identified. Applications will be illustrated by a complete study on nonlinear ordinary differential equations with either mixed or Dirichlet boundary conditions, in the context of phase transitions. The existence of smooth solutions will be discussed, and the iterative *Finite Difference Method (FDM)* is used to illustrate the difficulty of capturing non-smooth solutions with traditional methods. The results illustrate the important fact that smooth analytic or numerical solutions of a nonlinear mixed boundary-value problem might not be minimizers of the associated potential variational problem. From a *dual* perspective, the convergence (or non-convergence) of the FDM is explained and numerical examples are provided. The speaker will also show that some NP-hard problems in global optimization and computational science can be solved in an elegant way. The connection between some important physical phenomena and algebraic geometry will be revealed.

This talk should bring some new insights into nonconvex analysis, nonlinear PDEs, phase transitions, chaotic dynamics, and computational methods.