Description of adhering with saturation using boundary conditions of hysteresis type

N.D. Botkin[1], K.-H. Hoffmann[1], A.M. Meirmanov[2], V.N. Starovoitov[1]

The problem we consider in this paper appears when developing sensors which serve for detecting of certain proteins in solutions. An important part of such sensors is a wet cell, say a cube, filled with water into which a solution containing the protein to be detected is injected. Special molecules called aptamers are immobilized on the bottom of the wet cell. The aptamers can selectively bind the desired protein from the solution. The change of the surface mass loading can be analyzed using acoustic waves propagating along the aptamer layer. Thus, the concentration of the protein in the solution can be estimated. In this paper, a model that describes the propagation of the protein in the wet cell and its adhering to the aptamer is proposed. It is assumed for simplicity that the propagation of the injected protein in the wet cell is governed by an diffusion equation. A special boundary condition on the bottom provides the monotone grows of the deposited layer with saturation which means the exhaustion of free aptamer molecules.

The mathematical formulation looks as follows: $u_t = \Delta u$ in the interior of the wet cell, $\partial u/\partial \nu = -(\mathcal{A}(u))_t$ on the bottom of the wet cell. Here u is a scaled volume fraction of the protein so that it can be grater then one, $\mathcal{A}$ is a hysteresis operator defined by the relation: $\mathcal{A}(u)(t) = \operatorname{ess\,sup}\{H\big(u(\tau)\big) : \tau \leq t\}$. The function H is of the form: $H(s) = s/\varphi_0$, if $s \in [0, \varphi_0]$; $H(s) = 1$, if $s > \varphi_0$, where φ_0 is the saturation threshold. We prove the existence and the uniqueness of solutions to the problem, study the regularity of solutions, and perform numerical simulations that clarify the behavior of solutions.

References

[1] M. A. Krasnosel'skii, A. V. Pokrovskii. Systems with hysteresis. Springer-Verlag, Berlin, 1989, 410 p. (Russian edition: Nauka, Moscow, 1983).

[2] A. Visintin. Differential models of hysteresis. Applied Mathematical Sciences, V.111, Springer-Verlag, Berlin, 1994.

[3] M. Brokate, J. Sprekels. Hysteresis and phase transitions. Applied Mathematical Sciences, V.121, Springer-Verlag, New York, 1996.

[4] K.-H. Hoffmann, J. Sprekels.(eds.) Free Boundary Problems: Theory and Applications, Vol. I, II. Pitman Research Notes in Mathematics Series Vol. 185,186. Burnt Mill, Harlow: Longman.

[5] M. Hilpert. On uniqueness for evolution problems with hysteresis. In: Mathematical models for phase change problems. Internat. Ser. Numer. Math., V.88, Birkhäuser, Basel, 1989, pp.377–388.

[6] O. A. Ladyzhenskaya,V. A. Solonnikov, N. N. Ural'tseva. Linear and quasi-linear equations of parabolic type. Translations of Mathematical Monographs. 23. Providence, RI: American Mathematical Society. XI, 648 p. (1968).

[1]Center of Advanced European Studies and Research, Bonn, Germany.
[2]University of Novosibirsk, Russia.